map interchanging the factors. If $w \in H_n(X,\dot X;\, R)$ is a fundamental class of X, then $w \times w \in H_{2n}((X,\dot X) \times (X,\dot X);\, R)$ is a fundamental class of $X \times X$ (whence $X \times X$ is orientable), and $T_*(w \times w) = (-1)^n w \times w$. By theorem 9, T maps the orientation of $X \times X$ corresponding to $w \times w$ into $(-1)^n$ times itself. Let

$$\gamma\colon H_m(X \times X,\, X \times X - \delta(X);\, G) \approx \bar H^{2n-m}(\delta(X),\delta(\dot X);G)$$

be the duality map associated to this orientation. Then we have a commutative diagram (all coefficients G)

$$\begin{tikzcd}
H_m(X \times X,\, X \times X - \delta(X);\, G) \arrow[rr, "T_*"] \arrow[dr, "\gamma"', "\approx"] & & H_m(X \times X,\, X \times X - \delta(X);\, G) \arrow[dl, "(-1)^n\gamma", "\approx"'] \\
& \bar H^{2n-m}(\delta(X),\delta(\dot X);G) &
\end{tikzcd}$$

Therefore $T_*(z) = (-1)^n z$ for any $z \in H_*(X \times X,\, X \times X - \delta(X);\, G)$ (which implies $T^*(u) = (-1)^n u$ for any $u \in H^*(X \times X,\, X \times X - \delta(X);\, G)$). Then

$$p_{2*}(u \frown z) = p_{1*}T_*(u \frown z) = p_{1*}(T^*u \frown T_*z) = p_{1*}(u \frown z)$$

and $u \smile p_2^* v = (-1)^n T^*(u \smile p_2^* v) = u \smile T^* p_2^* v = u \smile p_1^* v$ ■

12 THEOREM *Let z be a fundamental class over R of a compact n-manifold X with boundary $\dot X$. For all q and R modules G the homomorphism $\kappa_z(v) = v \frown z$ defines isomorphisms*

$$\kappa_z\colon H^q(X;G) \approx H_{n-q}(X,\dot X;\, G)$$
$$\kappa_z\colon H^q(X,\dot X;\, G) \approx H_{n-q}(X;G)$$

which are, up to sign, the inverse of the duality isomorphisms of theorem 6.2.20 *defined by the orientation corresponding to z.*

PROOF Let U be the orientation of X corresponding to z as in theorem 9, and let $j\colon X - \dot X \subset X$. We prove commutativity up to sign in the triangle (all coefficients G)

$$\begin{tikzcd}
H_q(X - \dot X) \arrow[rr, "\gamma_U"] \arrow[dr, "j_*"'] & & H^{n-q}(X,\dot X) \arrow[dl, "\kappa_z"] \\
& H_q(X) &
\end{tikzcd}$$

For $w \in H_q(X - \dot X)$, by property 6.1.6,

$$\begin{aligned}
k_z\gamma_U(w) &= \{[U \mid (X,\dot X) \times (X - \dot X)]/w\} \frown z \\
&= p_{1*}\{[U \mid (X,\dot X) \times (X - \dot X)] \frown (z \times w)\}
\end{aligned}$$

By lemma 11, this equals

$$\begin{aligned}
&p_{2*}\{[U \mid (X,\dot X) \times (X - \dot X)] \frown (z \times w)\} \\
&\qquad = p_{1*}T_*\{[U \mid (X,\dot X) \times (X - \dot X)] \frown (z \times w)\} \\
&\qquad = \pm j_* \bar p_{1*}\{[U \mid (X - \dot X) \times (X,\dot X)] \frown (w \times z)\}
\end{aligned}$$

where $\bar p_1\colon (X - \dot X) \times X \to X - \dot X$ is projection to the first factor. Again by property 6.1.6,

$$\bar{p}_{1*}\{[U \mid (X - \dot{X}) \times (X,\dot{X})] \frown (w \times z)\} = \gamma_U(z) \frown w = w$$

Therefore

$$\kappa_z \gamma_U(w) = \pm j_*(w)$$

Similarly, we prove commutativity up to sign in the triangle

$$\begin{array}{ccc} H^q(X) & \xrightarrow{\kappa_z} & H_{n-q}(X,\dot{X}) \\ & {\scriptstyle j^*}\searrow \quad \swarrow{\scriptstyle \gamma_U} & \\ & H^q(X - \dot{X}) & \end{array}$$

For $v \in H^q(X)$, by property 6.1.5,

$$\begin{aligned} \gamma_U \kappa_z(v) &= [U \mid (X - \dot{X}) \times (X,\dot{X})]/(v \frown z) \\ &= \{[U \smile p_2^*(v)] \mid (X - \dot{X}) \times (X,\dot{X})\}/z \end{aligned}$$

By lemma 11 and property 6.1.4, this equals

$$\pm\{[\bar{p}_1^* j^*(v) \smile U] \mid (X - \dot{X}) \times (X,\dot{X})\}/z = \pm j^*(v) \smile \gamma_U(z) = \pm j^*(v)$$

Therefore

$$\gamma_U \kappa_z(v) = \pm j^* v \quad \blacksquare$$

4 THE ALEXANDER COHOMOLOGY THEORY

We shall now describe a cohomology theory particularly suited for applications in which a space is mapped into polyhedra (the singular theory is more suitable for applications where polyhedra are mapped into a space). One approach to the theory, called the Čech construction, is based on approximating a space by nerves of open coverings; another approach, called the Alexander-Kolmogoroff construction, is based on complexes built of "small" simplexes consisting of finite sets of points. We shall begin with the Alexander construction, and show later in the chapter (see corollary 6.9.9 and the following paragraph) that if (A,B) is a closed pair in a manifold X, then $\bar{H}^q(A,B; G)$ as defined in Sec. 6.1 is the Alexander cohomology of (A,B) with coefficients G.

Let G be an R module and let X be a topological space. For $q \geq 0$ let $C^q(X;G)$ be the module of all functions φ from X^{q+1} to G with addition and scalar multiplication defined pointwise. Thus, if $x_0, x_1, \ldots, x_q \in X$, then $\varphi(x_0,x_1, \ldots ,x_q) \in G$, and if $\varphi_1, \varphi_2 \in C^q(X;G)$ and $r \in R$, then

$$\begin{aligned} r\varphi_1(x_0, \ldots ,x_q) &= r(\varphi_1(x_0, \ldots ,x_q)) \\ (\varphi_1 + \varphi_2)(x_0, \ldots ,x_q) &= \varphi_1(x_0, \ldots ,x_q) + \varphi_2(x_0, \ldots ,x_q) \end{aligned}$$

We shall omit the symbol G from $C^q(X;G)$ where its absence will not cause confusion.

A coboundary homomorphism δ: $C^q(X) \to C^{q+1}(X)$ is defined by the formula

$$(\delta\varphi)(x_0, \ldots, x_{q+1}) = \sum_{0 \le i \le q+1} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1})$$

Then $\delta\delta = 0$ and $C^*(X) = \{C^q(X), \delta\}$ is a cochain complex over R. If X is nonempty, it is augmented over G by $\eta\colon G \to C^0(X)$, where $(\eta(g))(x) = g$ for $g \in G$ and all $x \in X$. So far the topology of X has played no role, and the following result shows that $C^*(X)$ has uninteresting cohomology.

1 LEMMA *If X is a nonempty space, $\eta^*\colon G \approx H^*(C^*(X;G))$.*

PROOF Let $\bar{x}$ be a fixed point of X and define a cochain homotopy $D\colon C^*(X) \to C^*(X)$ by

$$(D\varphi)(x_0, \ldots, x_q) = \varphi(\bar{x}, x_0, \ldots, x_q) \qquad q \ge 0$$

Then
$$\delta D\varphi + D\delta\varphi = \begin{cases} \varphi & \deg \varphi > 0 \\ \varphi - \eta(\varphi(\bar{x})) & \deg \varphi = 0 \end{cases}$$

Therefore, if $\tau\colon C(X;G) \to G$ is the cochain map defined by

$$\tau(\varphi) = \begin{cases} 0 & \deg \varphi > 0 \\ \varphi(\bar{x}) & \deg \varphi = 0 \end{cases}$$

then $\tau\eta = 1_G$ and D is a cochain homotopy from $1_{C^*(X)}$ to $\eta\tau$. Therefore η is a cochain equivalence, whence the result. ∎

We now use the topology of X to pass to a more interesting quotient complex. An element $\varphi \in C^q(X)$ is said to be *locally zero* if there is a covering $\mathcal{U}$ of X by open sets such that φ vanishes on any $(q+1)$-tuple of X which lies in some element of $\mathcal{U}$. Thus, if we define $\mathcal{U}^{q+1} = \bigcup_{U \in \mathcal{U}} U^{q+1} \subset X^{q+1}$, then φ vanishes on $\mathcal{U}^{q+1}$. The subset of $C^q(X)$ consisting of locally zero functions is a submodule, denoted by $C_0{}^q(X)$, and if φ vanishes on $\mathcal{U}^{q+1}$, then $\delta\varphi$ vanishes on $\mathcal{U}^{q+2}$, whence $C_0^*(X) = \{C_0{}^q(X), \delta\}$ is a cochain subcomplex of $C^*(X)$. We define $\bar{C}^*(X)$ to be the quotient cochain complex of $C^*(X)$ by $C_0^*(X)$. If X is nonempty, the composite

$$G \xrightarrow{\eta} C^*(X) \to \bar{C}^*(X)$$

is an augmentation of $\bar{C}^*(X)$, also denoted by η. The cohomology module of $\bar{C}^*(X)$ of degree q is denoted by $\bar{H}^q(X;G)$.

Given a function $f\colon X \to Y$ (not necessarily continuous), there is an induced cochain map

$$f^\#\colon C^*(Y;G) \to C^*(X;G)$$

defined by the formula

$$(f^\#\varphi)(x_0, \ldots, x_q) = \varphi(f(x_0), \ldots, f(x_q)) \qquad \varphi \in C^q(Y);\ x_0, \ldots, x_q \in X$$

If φ vanishes on $\mathcal{V}^{q+1}$, where $\mathcal{V}$ is an open covering of Y, and if there is an open covering $\mathcal{U}$ of X such that f maps each element of $\mathcal{U}$ into some element of $\mathcal{V}$, then $f^\#\varphi$ vanishes on $\mathcal{U}^{q+1}$. In particular, if f is continuous, $f^{-1}\mathcal{V}$ is an open covering of X which can be taken as $\mathcal{U}$, and therefore $f^\#$

maps $C_0^*(Y)$ into $C_0^*(X)$. It follows that if f is continuous, there is an induced cochain map

$$f\#: \bar{C}^*(Y;G) \to \bar{C}^*(X;G)$$

Let A be a subspace of X and let $i: A \subset X$. Then $i\#: \bar{C}^*(X;G) \to \bar{C}^*(A;G)$ is an epimorphism. Therefore the kernel of $i\#$ is a cochain subcomplex of $\bar{C}^*(X;G)$, denoted by $\bar{C}^*(X,A;\ G)$. The relative module $\bar{H}^q(X,A;\ G)$ is defined to be the cohomology module of $\bar{C}^*(X,A;\ G)$ of degree q.

Since there is a short exact sequence of cochain complexes

$$0 \to \bar{C}^*(X,A;\ G) \xrightarrow{j\#} \bar{C}^*(X;G) \xrightarrow{i\#} \bar{C}^*(A;G) \to 0$$

it follows that there is an exact sequence

2 $$\cdots \to \bar{H}^q(X,A;\ G) \xrightarrow{j^*} \bar{H}^q(X;G) \xrightarrow{i^*} \bar{H}^q(A;G) \xrightarrow{\delta^*} \bar{H}^{q+1}(X,A;\ G) \to \cdots$$

The graded module $\bar{H}^*(X,A) = \{\bar{H}^q(X,A;\ G)\}$ is the module function of the cohomology theory we are constructing, and the homomorphism $\delta^*: \bar{H}^q(A;G) \to \bar{H}^{q+1}(X,A;\ G)$ is the connecting homomorphism of the theory. Given a continuous map $f: (X,A) \to (Y,B)$, there is induced by f a commutative diagram of cochain maps

$$\begin{array}{ccccccccc}
0 & \to & \bar{C}^*(Y,B;\ G) & \to & \bar{C}^*(Y;G) & \to & \bar{C}^*(B;G) & \to & 0 \\
 & & {\scriptstyle f\#}\downarrow & & {\scriptstyle (f|X)\#}\downarrow & & \downarrow{\scriptstyle (f|A)\#} & & \\
0 & \to & \bar{C}^*(X,A;\ G) & \to & \bar{C}^*(X;G) & \to & \bar{C}^*(A;G) & \to & 0
\end{array}$$

The homomorphism $f^*: \bar{H}^*(Y,B;\ G) \to \bar{H}^*(X,A;\ G)$ is defined to be the homomorphism induced by the cochain map $f\#$ in the above diagram. It is then clear that for fixed G, $\bar{H}^*(X,A;\ G)$ and f^* constitute a contravariant functor from the category of topological pairs to the category of graded R modules. Furthermore, the connecting homomorphism δ^* is a natural transformation of degree 1 from $\bar{H}^*(A;G)$ to $\bar{H}^*(X,A;\ G)$. Therefore we have the constituents of a cohomology theory, and we shall verify that the axioms are satisfied. The resulting cohomology theory is called the *Alexander* (or *Alexander-Spanier*[1]) *cohomology theory*, and $\bar{H}^q(X,A;\ G)$ is called the *Alexander cohomology module of* (X,A) *of degree* q *with coefficients* G.

The exactness axiom is a consequence of the exactness of the sequence 2. The dimension axiom will follow from the next result.

3 LEMMA *If X is a one-point space, $\eta^*: G \approx \bar{H}^*(X;G)$.*

PROOF Because X is a one-point space, a locally zero function on X is zero. Therefore $\bar{C}^*(X;G) = C^*(X;G)$ and the result follows from lemma 1. ∎

Before proving the excision axiom it will be useful to introduce another cochain complex for the relative theory. If $A \subset X$, let $C^*(X,A)$ be the sub-

[1] See E. Spanier, Cohomology theory for general spaces, *Annals of Mathematics*, vol. 49 pp. 407–427, 1948.

complex of $C^*(X)$ of functions φ which are locally zero on A. Thus there is a short exact sequence

$$0 \to C^*(X,A) \to C^*(X) \to \bar{C}^*(A) \to 0$$

and $C_0^*(X) \subset C^*(X,A)$. It follows that $\bar{C}^*(X,A) = C^*(X,A)/C_0^*(X)$. The excision axiom follows from the next result.

4 LEMMA *Let U be a subset of $A \subset X$ such that U has an open neighborhood W with $\bar{W} \subset \operatorname{int} A$. Then the inclusion map j: $(X - U, A - U) \subset (X,A)$ induces an isomorphism*

$$j^\#: \bar{C}^*(X,A) \approx \bar{C}^*(X - U, A - U)$$

PROOF There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & C_0^*(X) & \to & C^*(X,A) & \to & \bar{C}^*(X,A) & \to 0 \\
 & \downarrow & & \downarrow k^\# & & \downarrow j^\# & \\
0 \to & C_0^*(X - U) & \to & C^*(X - U, A - U) & \xrightarrow{\lambda} & \bar{C}^*(X - U, A - U) & \to 0
\end{array}$$

It suffices to prove that $\lambda k^\#$ is an epimorphism and that $(k^\#)^{-1}(C_0^*(X - U)) = C_0^*(X)$. If $\varphi \in C^q(X - U, A - U)$, let $\bar{\varphi} \in C^q(X)$ be defined by

$$\bar{\varphi}(x_0, \ldots, x_q) = \begin{cases} 0 & x_i \in W \text{ for some } 0 \le i \le q \\ \varphi(x_0, \ldots, x_q) & x_0, \ldots, x_q \in X - W \end{cases}$$

If $\mathcal{V}$ is an open covering of $A - U$ such that φ vanishes on $\mathcal{V}^{q+1}$, then $\mathcal{U} = \{V \cup W \mid V \in \mathcal{V}\}$ is an open covering of A such that $\bar{\varphi}$ vanishes on $\mathcal{U}^{q+1}$. Therefore $\bar{\varphi} \in C^q(X,A)$, and from the definition of $\bar{\varphi}$, $k^\#\bar{\varphi} - \varphi$ vanishes on $\mathcal{W}^{q+1}$ where $\mathcal{W} = \{V \cap \operatorname{int} A \mid V \in \mathcal{V}\} \cup \{X - \bar{W}\}$, which is an open covering of $X - U$. Therefore $\lambda k^\# \bar{\varphi} = \lambda\varphi$, and because λ is an epimorphism, so is $\lambda k^\#$.

Assume that $\varphi \in C^q(X,A)$ is such that $k^\#\varphi \in C_0{}^q(X - U)$. Because φ is locally zero on A, there is an open covering $\mathcal{U}_1$ of A such that φ vanishes on $\mathcal{U}_1{}^{q+1}$. Because $k^\#\varphi \in C_0{}^q(X - U)$, there is an open covering $\mathcal{U}_2$ of $X - U$ such that φ vanishes on $\mathcal{U}_2{}^{q+1}$. Let

$$\mathcal{V}_1 = \{U_1 \cap \operatorname{int} A \mid U_1 \in \mathcal{U}_1\} \qquad \mathcal{V}_2 = \{U_2 \cap (X - \bar{U}) \mid U_2 \in \mathcal{U}_2\}$$

Then $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is an open covering of X such that φ vanishes on $\mathcal{V}^{q+1}$. Therefore $\varphi \in C_0{}^q(X)$ and so

$$(k^\#)^{-1}(C_0^*(X - U)) = C_0^*(X) \quad ■$$

The homotopy axiom will be proved in the next section. We conclude this section with a study of $\bar{H}^0$. A function φ from a topological space X to a set is said to be *locally constant* if there is an open covering $\mathcal{U}$ of X such that φ is constant on each element of $\mathcal{U}$.

5 THEOREM *If $A \subset X$, then $\bar{H}^0(X,A; G)$ is isomorphic to the module of locally constant functions from X to G which vanish on A.*

PROOF A locally zero function from X to G is zero. Therefore $C_0{}^0(X) = 0$, and so

$$\bar{C}^0(X,A) = C^0(X,A)/C_0{}^0(X) = C^0(X,A)$$

Therefore $\bar{H}^0(X,A;\, G)$ is the kernel of the composite

$$C^0(X,A) \xrightarrow{\delta} C^1(X,A) \to \bar{C}^1(X,A)$$

$C^0(X,A)$ is the module of functions from X to G which vanish on A. If $\varphi \in C^0(X,A)$, then φ is in the kernel of the above composite if and only if there is some open covering $\mathfrak{U}$ of X such that $\delta\varphi$ vanishes on $\mathfrak{U}^2$. Since $(\delta\varphi)(x,y) = \varphi(y) - \varphi(x)$, this is equivalent to the condition that there is an open covering $\mathfrak{U}$ such that φ is constant on each element of $\mathfrak{U}$. Hence the kernel of the above composite equals the module of functions vanishing on A that are locally constant on X. ■

6 **COROLLARY** *Let X be a topological space in which every quasi-component is open and let $A \subset X$. Then $\bar{H}^0(X,A;\, G)$ is isomorphic to the module of functions from the set of those quasi-components of X which do not intersect A to G.*

PROOF This follows from theorem 5 and the fact that a locally constant function on X is constant on every quasi-component of X. ■

7 **COROLLARY** *A nonempty space X is connected if and only if*

$$\eta^*\colon G \approx \bar{H}^0(X;G)$$

PROOF This follows from theorem 5 and the trivial observation that every locally constant function on X is constant if and only if X is connected. ■

It follows that there exist spaces for which the singular cohomology and Alexander cohomology differ. In fact, for any connected space which is not path connected, corollary 7 and theorem 5.4.10 show that they differ in degree 0.

We now present a version of theorem 5.4.10 valid for the Alexander theory.

8 **THEOREM** *Let $\{U_j\}$ be an open covering of X by pairwise disjoint sets. Then there is a canonical isomorphism*

$$\bar{H}^q(X;G) \approx \times\, \bar{H}^q(U_j;G)$$

PROOF Because $\{U_j\}$ consists of pairwise disjoint sets, the map induced by restriction

$$i^\#\colon C^*(X) \to \times\, C^*(U_j)$$

is an epimorphism. Because $\{U_j\}$ is an open covering of X, it follows that

$$(i^\#)^{-1}(\times\, C_0^*(U_j)) = C_0^*(X)$$

Therefore $i^\#$ induces an isomorphism $\bar{C}^*(X) \approx \times\, \bar{C}^*(U_j)$. ■

9 COROLLARY *Let $\{C_j\}$ be the collection of components of a locally connected space X. Then there is a canonical isomorphism*

$$\bar{H}^q(X;G) \approx \times \bar{H}^q(C_j;G)$$

PROOF Because X is locally connected, its components are open, and the result follows from theorem 8. ∎

5 THE HOMOTOPY AXIOM FOR THE ALEXANDER THEORY

In this section we shall prove the homotopy axiom for the Alexander cohomology theory. The proof will be based on a description of the Alexander cochain complex as the limit of cochain complexes of abstract simplicial complexes. We shall also use this description to construct a homomorphism of the Alexander cohomology theory into the singular cohomology theory. Because the Alexander theory satisfies all the axioms, this homomorphism is an isomorphism from the Alexander theory to the singular cohomology theory on the category of compact polyhedral pairs.

We shall be considering a fixed R module G as coefficient module for cohomology and will usually not mention G explicitly. Let $\mathcal{U}$ be a collection of subsets covering a set X. Let $X(\mathcal{U})$ be the abstract simplicial complex whose vertices are the points of X and whose simplexes are finite subsets F of X such that there is some $U \in \mathcal{U}$ containing F. Let $C(\mathcal{U})$ be the ordered chain complex of $X(\mathcal{U})$ over R. Given a subset $A \subset X$ and a subcollection $\mathcal{U}' \subset \mathcal{U}$ which covers A, we let $A(\mathcal{U}')$ be the subcomplex of $X(\mathcal{U})$ whose vertices are the points of A and whose simplexes are finite subsets of A lying in some element of $\mathcal{U}'$. Then $C'(\mathcal{U}')$ will denote the chain subcomplex of $C(\mathcal{U})$ corresponding to $A(\mathcal{U}')$.

Let $(\mathcal{V},\mathcal{V}')$ be another pair consisting of a covering $\mathcal{V}$ of X and a subset $\mathcal{V}' \subset \mathcal{V}$ which is a covering of A. Assume that $(\mathcal{V},\mathcal{V}')$ is a refinement of $(\mathcal{U},\mathcal{U}')$ in the sense that every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$ and every element of $\mathcal{V}'$ is contained in some element of $\mathcal{U}'$. Then the pair $(C(\mathcal{V}),C'(\mathcal{V}'))$ is mapped injectively into the pair $(C(\mathcal{U}),C'(\mathcal{U}'))$ by the identity map of (X,A) to itself.

Let X be a topological space and A a subspace of X. Consider pairs $(\mathcal{U},\mathcal{U}')$, where $\mathcal{U}$ is an open covering of X and $\mathcal{U}'$ is a subset of $\mathcal{U}$ which covers A. Such a pair is called an *open covering* of (X,A). Let $C^*(\mathcal{U},\mathcal{U}')$ be the cochain complex of the pair $(C(\mathcal{U}),C'(\mathcal{U}'))$ (with coefficients in G). An element u of $C^q(\mathcal{U},\mathcal{U}')$ is a function defined on $(q + 1)$-tuples of X which lie in some element of $\mathcal{U}$, taking values in G, and vanishing on $(q + 1)$-tuples of A which lie in some element of $\mathcal{U}'$. If $(\mathcal{V},\mathcal{V}')$ is a refinement of $(\mathcal{U},\mathcal{U}')$, the restriction map is a cochain map

$$C^*(\mathcal{U},\mathcal{U}') \to C^*(\mathcal{V},\mathcal{V}')$$

If $(\mathcal{U},\mathcal{U}')$ and $(\mathcal{V},\mathcal{V}')$ are two open coverings of (X,A) as above, let $\mathcal{W} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ and let $\mathcal{W}' = \{U' \cap V' \mid U' \in \mathcal{U}', V' \in \mathcal{V}'\}$. Then $(\mathcal{W},\mathcal{W}')$ is another open covering of (X,A) and $(\mathcal{W},\mathcal{W}')$ is a refinement of $(\mathcal{U},\mathcal{U}')$ and of $(\mathcal{V},\mathcal{V}')$. Therefore the cochain complexes $\{C^*(\mathcal{U},\mathcal{U}')\}$ form a direct system, and we have a limit cochain complex

$$\lim_{\rightarrow} \{C^*(\mathcal{U},\mathcal{U}')\}$$

We shall show that this limit cochain complex is canonically isomorphic to $\bar{C}^*(X,A)$. If $\varphi \in C^q(X,A)$, let $\mathcal{U}'$ be a collection of open subsets of X covering A such that φ vanishes on $(\mathcal{U}')^{q+1} \cap A^{q+1}$ (such a $\mathcal{U}'$ exists because φ is locally zero) and let $\mathcal{U} = \mathcal{U}' \cup \{X\}$. Then $(\mathcal{U},\mathcal{U}')$ is an open covering of (X,A) and φ determines by restriction an element $\varphi \mid (\mathcal{U},\mathcal{U}') \in C^q(\mathcal{U},\mathcal{U}')$. Passing to the limit, we obtain a homomorphism (by restriction)

$$\lambda: C^*(X,A) \to \lim_{\rightarrow} \{C^*(\mathcal{U},\mathcal{U}')\}$$

which is a canonical cochain map. The following result explains our interest in the cochain complexes $C^*(\mathcal{U},\mathcal{U}')$.

1 THEOREM *The canonical cochain map*

$$\lambda: C^*(X,A) \to \lim_{\rightarrow} \{C^*(\mathcal{U},\mathcal{U}')\}$$

is an epimorphism and has kernel equal to $C_0^*(X)$.

PROOF To prove that λ is an epimorphism, let $u \in C^q(\mathcal{U},\mathcal{U}')$. Define $\varphi_u \in C^q(X)$ by

$$\varphi_u(x_0, \ldots ,x_q) = \begin{cases} u(x_0, \ldots ,x_q) & \text{if } x_0, \ldots , x_q \in U, \text{ where } U \in \mathcal{U} \\ 0 & \text{otherwise} \end{cases}$$

Then φ_u vanishes on $(\mathcal{U}')^{q+1} \cap A^{q+1}$, and therefore $\varphi_u \in C^q(X,A)$. By definition, $\varphi_u \mid (\mathcal{U},\mathcal{U}') = u$, and λ is an epimorphism.

An element $\varphi \in C^q(X,A)$ is in the kernel of λ if and only if there is some $(\mathcal{U},\mathcal{U}')$ such that $\varphi \mid (\mathcal{U},\mathcal{U}') = 0$. Thus $\lambda(\varphi) = 0$ if and only if there is some open covering $\mathcal{U}$ such that φ vanishes on $\mathcal{U}^{q+1}$. By the definition of $C_0^*(X)$, $\lambda(\varphi) = 0$ if and only if $\varphi \in C_0^*(X)$. ∎

From theorem 1 and the analogue of theorem 4.1.7 for cochain complexes, we have the following corollary.

2 COROLLARY *For the Alexander cohomology theory there is a canonical isomorphism*

$$\bar{H}^q(X,A;G) \approx \lim_{\rightarrow} \{H^q(C^*(\mathcal{U},\mathcal{U}';G))\} \quad ∎$$

We are now ready for the proof of the homotopy axiom for the Alexander cohomology theory. In the presence of the other axioms, it suffices to prove it for the case of the two mappings

$$h_0, h_1: (X,A) \to (X \times I, A \times I)$$

where $h_0(x) = (x,0)$, $h_1(x) = (x,1)$. The proof consists in showing that if $(\mathcal{U},\mathcal{U}')$ is any open covering of $(X \times I, A \times I)$, there is an open covering $(\mathcal{V},\mathcal{V}')$ of (X,A) such that h_0 and h_1 induce chain-homotopic chain maps from $(C(\mathcal{V}),C(\mathcal{V}'))$ to $(C(\mathcal{U}),C(\mathcal{U}'))$. This is a result about free chain complexes, and the technique of acyclic models is available for obtaining the desired chain homotopy.

Let Y be an arbitrary set and n a nonnegative integer. Let $C(Y,n)$ be the chain complex over R of the abstract simplicial complex $(Y \times I)(\mathcal{U}(Y,n))$, where $\mathcal{U}(Y,n)$ is the covering of $Y \times I$ defined by

$$\mathcal{U}(Y,n) = \left\{ Y \times \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right] \,\middle|\, 0 \le m < 2^n \right\}$$

3 LEMMA *If Y is nonempty, the chain complex $\tilde{C}(Y,n)$ is acyclic.*

PROOF For $0 \le m < 2^n$ let K_m be the subcomplex of $(Y \times I)(\mathcal{U}(Y,n))$ consisting of all the finite subsets of $Y \times [m/2^n, (m+1)/2^n]$. For $0 \le m \le 2^n$ let L_m be the subcomplex of $(Y \times I)(\mathcal{U}(Y,n))$ consisting of all the finite subsets of $Y \times (m/2^n)$. Then $(Y \times I)(\mathcal{U}(Y,n)) = \bigcup_m K_m$ and $K_i \cap K_j = 0$ if $|i - j| > 1$ and $K_i \cap K_{i+1} = L_{i+1}$. Because Y is nonempty, each K_m (and L_m) is nonempty and is the join of K_m (or L_m) with any vertex in it. Therefore, by theorem 4.3.6, $\tilde{C}(K_m)$ and $\tilde{C}(L_m)$ are acyclic. Let $N_q = \bigcup_{m \le q} K_m$. Then $N_{q+1} = N_q \cup K_{q+1}$ and $N_q \cap K_{q+1} = L_{q+1}$. By induction on q, using the exactness of the reduced Mayer-Vietoris sequence, it follows that $\tilde{C}(N_q)$ is acyclic for all q. Therefore $\tilde{C}(Y,n) = \tilde{C}(N_{2^n-1})$ is acyclic. ■

From this we have our next result, which will provide the acyclic model for the homotopy axiom.

4 LEMMA *Let $Y_1, \ldots, Y_q$ be subsets of a nonempty set Y, where $Y = Y_1$, and for each i let n_i be a nonnegative integer. Let K be the simplicial complex defined by*

$$K = \bigcup_i (Y_i \times I)(\mathcal{U}(Y_i,n_i))$$

Then $\tilde{C}(K)$ is acyclic.

PROOF We prove the lemma by induction on q. If $q = 1$, it follows from lemma 3. Assume that $q > 1$, and the result is valid for fewer than q sets Y_i. Let $\bar{K} = \bigcup_{i \le q-1} (Y_i \times I)(\mathcal{U}(Y_i,n_i))$. Then $\bar{K} \cup (Y_q \times I)(\mathcal{U}(Y_q,n_q)) = K$. If Y_q is empty, $\tilde{C}(K) = \tilde{C}(\bar{K})$ is acyclic, by the inductive assumption. If Y_q is nonempty, $\tilde{C}(Y_q,n_q)$ is acyclic, by lemma 3, and $\tilde{C}(\bar{K})$ is acyclic, by the inductive assumption. To prove that $\tilde{C}(K)$ is acyclic, from the exactness of the reduced Mayer-Vietoris sequence it suffices to prove that $\tilde{C}(\bar{K} \cap (Y_q \times I)(\mathcal{U}(Y_q,n_q)))$ is acyclic. However, $\bar{K} \cap (Y_q \times I)(\mathcal{U}(Y_q,n_q)) = \bigcup_{1 \le i < q} (Y_i' \times I)(\mathcal{U}(Y_i',n_i'))$, where $Y_i' = Y_i \cap Y_q$ are subsets of Y_q (and $Y_1' = Y_q$) and $n_i' = \max(n_i,n_q)$. Therefore, by the inductive assumption, $\tilde{C}(\bar{K} \cap (Y_q \times I)(\mathcal{U}(Y_q,n_q)))$ is acyclic. ■

We now come to the following main step in the proof of the homotopy axiom.

5 LEMMA *Let $(\mathcal{U},\mathcal{U}')$ be any open covering of $(X \times I, A \times I)$. There is an open covering $(\mathcal{V},\mathcal{V}')$ of (X,A) such that h_0 and h_1 induce chain-homotopic chain maps from $(C(\mathcal{V}),C'(\mathcal{V}'))$ to $(C(\mathcal{U}),C'(\mathcal{U}'))$.*

PROOF For each $x \in X$ it follows from the compactness of $x \times I$ that there is an open set V_x about x and an integer $n \geq 0$ such that for $0 \leq m < 2^n$ the set $V_x \times [m/2^n, (m+1)/2^n]$ is contained in some element of $\mathcal{U}$. Furthermore, if $x \in A$, we can choose V_x and n so that $V_x \times [m/2^n, (m+1)/2^n]$ is contained in some element of $\mathcal{U}'$. Let $\mathcal{V}$ be the collection $\{V_x\}_{x\in X}$ and $\mathcal{V}'$ the subcollection $\{V_x\}_{x\in A}$. To show that $(\mathcal{V},\mathcal{V}')$ has the desired property, let $\mathcal{C}$ be the category consisting of the subcomplexes of $X(\mathcal{V})$ partially ordered by inclusion. For each subcomplex K of $X(\mathcal{V})$ let $G(K)$ be the ordered chain complex of K. For each simplex s of $X(\mathcal{V})$ [or $A(\mathcal{V}')$] define $n(s)$ to be the smallest nonnegative integer such that for $0 \leq m < n(s)$ each set $s \times [m/2^{n(s)}, (m+1)/2^{n(s)}]$ is contained in some element of $\mathcal{U}$ [or $\mathcal{U}'$]. Such an integer exists because of the way $(\mathcal{V},\mathcal{V}')$ was chosen. For a subcomplex K of $X(\mathcal{V})$ let $\hat{K}$ be the subcomplex of $(X \times I)(\mathcal{U})$ defined by

$$\hat{K} = \cup \{(s \times I)(\mathcal{U}(s,n(s)) \mid s \in K\}$$

and let $G'(K)$ be the ordered chain complex of $\hat{K}$. Then G and G' are covariant functors from $\mathcal{C}$ to the category of augmented chain complexes.

Let $\mathfrak{M}$ be the set of subcomplexes $\{\bar{s} \subset X(\mathcal{V}) \mid s \in X(\mathcal{V})\}$. Then G is free on $\mathcal{C}$ with models $\mathfrak{M}$, and by lemma 4, G' is acyclic on $\mathcal{C}$ with models $\mathfrak{M}$. If $\sigma = (x_0,x_1, \ldots ,x_q)$ is an ordered q-simplex of $X(\mathcal{V})$, then

$$h_0(\sigma) = ((x_0,0), \ldots ,(x_q,0)) \qquad \text{and} \qquad h_1(\sigma) = ((x_0,1), \ldots ,(x_q,1))$$

are natural chain maps preserving augmentation from G to G'. It follows from theorem 4.3.3 that there is a natural chain homotopy from h_0 to h_1. ∎

If $u \in H^q(C^*(\mathcal{U},\mathcal{U}'))$, where $(\mathcal{U},\mathcal{U}')$ is an open covering of $(X \times I, A \times I)$, it follows from lemma 5 that there is an open covering $(\mathcal{V},\mathcal{V}')$ of (X,A) such that $h_0(\mathcal{V},\mathcal{V}') \subset (\mathcal{U},\mathcal{U}')$, $h_1(\mathcal{V},\mathcal{V}') \subset (\mathcal{U},\mathcal{U}')$, and $h_0^* u = h_1^* u$ in $H^q(C^*(\mathcal{V},\mathcal{V}'))$. Passing to the limit and using corollary 2 gives us the final result.

6 THEOREM *The Alexander cohomology theory satisfies the homotopy axiom.* ∎

We have now verified all the axioms of cohomology theory for the Alexander cohomology theory. We construct a homomorphism μ from the Alexander cohomology theory to the singular cohomology theory. Let $(\mathcal{U},\mathcal{U}')$ be an open covering of (X,A). There is a canonical chain transformation

$$(\Delta(\mathcal{U}), \Delta(\mathcal{U}' \cap A)) \to (C(\mathcal{U}),C'(\mathcal{U}'))$$

which assigns to a singular q-simplex $\sigma: \Delta^q \to X$ the ordered simplex $(\sigma(v_0),\sigma(v_1), \ldots ,\sigma(v_q))$ of $C(\mathcal{U})$. This induces a homomorphism

$$C^*(\mathfrak{U},\mathfrak{U}'; G) \to C^*(\Delta(\mathfrak{U}), \Delta(\mathfrak{U}' \cap A); G)$$

Passing to the limit and using corollary 2, we obtain a canonical homomorphism

$$\mu': \bar{H}^q(X,A; G) \to \lim_{\to} \{H^q(\Delta(\mathfrak{U}), \Delta(\mathfrak{U}' \cap A); G)\}$$

By theorem 4.4.14, there is a canonical isomorphism

$$\mu'': H^q(\Delta(X), \Delta(A); G) \approx \lim_{\to} \{H^q(\Delta(\mathfrak{U}), \Delta(\mathfrak{U}' \cap A); G)\}$$

and the homomorphism

$$\mu: \bar{H}^q(X,A; G) \to H^q(\Delta(X), \Delta(A); G)$$

is defined to equal the composite $\mu''^{-1}\mu'$. It can be verified that this homomorphism has the commutativity properties necessary to be a natural transformation of cohomology theories.

We now introduce a cup product in the Alexander theory, which will have the usual properties of a cup product (as in Sec. 5.6) and will be compatible with the singular cup product by the homomorphism μ.

Let G and G' be R modules paired to an R module G''. Given $\varphi_1 \in C^p(X;G)$ and $\varphi_2 \in C^q(X;G')$, we define $\varphi_1 \smile \varphi_2 \in C^{p+q}(X;G'')$ by

$$(\varphi_1 \smile \varphi_2)(x_0, \ldots ,x_{p+q}) = \varphi_1(x_0, \ldots ,x_p)\varphi_2(x_p, \ldots ,x_{p+q})$$

If φ_1 is locally zero on A_1, so is $\varphi_1 \smile \varphi_2$, and if φ_2 is locally zero on A_2, so is $\varphi_1 \smile \varphi_2$. Therefore $\varphi_1 \smile \varphi_2$ induces a cup product from $\bar{C}^p(X;G)$ and $\bar{C}^q(X;G')$ to $\bar{C}^{p+q}(X;G'')$. An easy verification shows that

$$\delta(\varphi_1 \smile \varphi_2) = \delta\varphi_1 \smile \varphi_2 + (-1)^p\varphi_1 \smile \delta\varphi_2$$

Therefore the cup product induces a cup product on cohomology classes, and this cup product is clearly mapped by μ to the singular cup product.

In order to get a cup product from $C^p(X,A_1; G)$ and $C^q(X,A_2; G')$ to $C^{p+q}(X, A_1 \cup A_2; G'')$, we need to ensure that an element of $C^{p+q}(X;G'')$ which is locally zero on A_1 and locally zero on A_2 will be locally zero on $A_1 \cup A_2$. If $A_1 \cup A_2 = \operatorname{int}_{A_1 \cup A_2} A_1 \cup \operatorname{int}_{A_1 \cup A_2} A_2$, this is so. With this modification properties 5.6.8 to 5.6.12 are all valid for the resulting cohomology product.

6 TAUTNESS AND CONTINUITY

In this section we shall consider tautness for the Alexander theory and establish the strong result that any paracompact space imbedded as a closed subspace of a paracompact space is tautly imbedded. This implies a strong excision property for paracompact pairs (X,A) with A closed in X. It also implies the continuity property (that the Alexander cohomology theory commutes with limits of compact Hausdorff spaces directed by inclusion). This continuity property, together with the other axioms of cohomology theory, characterizes the Alexander theory on the category of compact Hausdorff pairs (that is, pairs with X compact Hausdorff and A closed in X). The section closes with a brief discussion of the Alexander cohomology with compact supports. Our proof of the special tautness properties of the Alexander cohomology theory is based on techniques of Wallace.[1]

Let $\mathcal{U}$ be a collection of subsets of a set X. Let $\mathcal{U}^* = \{U^*\}_{U \in \mathcal{U}}$, where

$$U^* = \cup \{U' \in \mathcal{U} \mid U' \cap U \neq \phi\}$$

A collection $\mathcal{V}$ is said to be a *star refinement* of $\mathcal{U}$ if $\mathcal{V}^*$ is a refinement of $\mathcal{U}$. A topological space X is said to be *fully normal* if every open covering of X has an open star refinement. It is known that for Hausdorff spaces paracompactness is equivalent to full normality.

1 LEMMA *Let A be a subset of a topological space X and let $\mathcal{V}$ be an open covering of X. There exist a neighborhood N of A and a function $f\colon N \to A$ (not necessarily continuous) such that*

(a) $f(x) = x$ for $x \in A$.
(b) If $V \in \mathcal{V}$, then $f(V \cap N) \subset V^*$.

PROOF If A is empty, let $N = A$ and f be the identity map. If A is nonempty, let $N = \cup \{V \in \mathcal{V} \mid V \cap A \neq \phi\}$ and define $f\colon N \to A$ by $f(x) = x$ for $x \in A$, or if $x \notin A$, choose $f(x) \in A$ so that there is $V \in \mathcal{V}$ with $x, f(x) \in V$. Such a choice of $f(x)$ is always possible because of the way N was defined. Clearly, if $x \in V \cap N$, there is $V' \in \mathcal{V}$ with x, $f(x) \in V'$. Therefore $x \in V \cap V'$ and $V' \subset V^*$. Hence, $f(V \cap N) \subset V^*$ and (a) and (b) are satisfied. ∎

This last result may be interpreted as asserting that A is a discontinuous neighborhood retract of X with a retraction that is not too discontinuous. If A is a closed subset of a paracompact space, it is similar enough to an absolute neighborhood retract so that we have the following generalization of theorem 6.1.10 for the Alexander theory.

2 THEOREM *A closed subspace of a paracompact Hausdorff space is a taut subspace relative to the Alexander cohomology theory.*

PROOF Let A be a closed subspace of a paracompact space X and let $\varphi \in C^q(A)$ be a cochain such that $\delta\varphi$ vanishes on $\mathcal{W}^{q+2}$, where $\mathcal{W}$ is an open covering of A. Let $\mathcal{U} = \{W \cup (X - A) \mid W \in \mathcal{W}\}$ and observe that $\mathcal{U}$ is an open covering of X because A is closed in X. Let $\mathcal{V}$ be an open star refinement of $\mathcal{U}$ and let N be a neighborhood of A and $f\colon N \to A$ a function (not necessarily continuous) satisfying lemma 1 relative to $\mathcal{V}$. Then $f^\#\varphi \in C^q(N)$, and we show that $\delta f^\#\varphi = f^\#\delta\varphi$ vanishes on $\mathcal{V}^{q+2} \cap N^{q+2}$. By lemma 1*b*, for any $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $f(V \cap N) \subset U$. Then $f(V \cap N) \subset U \cap A \subset W$ for some $W \in \mathcal{W}$. Therefore $\delta f^\#\varphi$ vanishes on $(V \cap N)^{q+2}$. This means that $f^\#\varphi$ represents a cocycle of $\bar{C}^q(N)$ and, by lemma 1*a*, $(f^\#\varphi) \mid A = \varphi$. Hence

[1] See A. D. Wallace, The map excision theorem, *Duke Mathematical Journal*, vol. 19, pp. 177–182, 1952.

the cohomology class $\{\varphi\} \in \bar{H}^q(A)$ is the image under restriction of the cohomology class $\{f^\#\varphi\} \in \bar{H}^q(N)$, showing that $\lim_\rightarrow \{\bar{H}^q(N)\} \to \bar{H}^q(A)$ is an epimorphism.

To prove that it is a monomorphism, let N' be a paracompact neighborhood of A and assume that $\varphi \in C^q(N')$ is such that $\delta\varphi$ vanishes on $\mathcal{W}^{q+2}$ and $\varphi \mid A = \delta\varphi'$ on $(\mathcal{W}')^{q+1}$, where $\mathcal{W}$ is an open covering of N' and $\mathcal{W}'$ is an open covering of A. Let $\mathcal{U} = \{W' \cup (N' - A) \mid W' \in \mathcal{W}'\}$ and observe that $\mathcal{U}$ is an open covering of N' (because A is closed.) Let $\mathcal{V}$ be an open star refinement of both $\mathcal{W}$ and $\mathcal{U}$ ($\mathcal{V}$ is a covering of N') and let N be a neighborhood of A in N' and $f\colon N \to A$ a function (not necessarily continuous) defined with respect to $\mathcal{V}$ to satisfy lemma 1. If $V \in \mathcal{V}$, then $f(V \cap N) \subset W'$ for some $W' \in \mathcal{W}'$. Therefore $f^\#(\varphi \mid A) = \delta f^\#\varphi'$ on $V^{q+1} \cap N^{q+1}$.

To show that $f^\#(\varphi \mid A)$ is cohomologous in $C^q(N)$ to $\varphi \mid N$, for $\psi \in C^p(N)$ define $D\psi \in C^{p-1}(N)$ by

$$(D\psi)(x_0, \ldots ,x_{p-1}) = \sum_{0 \le j \le p-1} (-1)^j \psi(x_0, \ldots ,x_j, f(x_j), \ldots ,f(x_{p-1}))$$

An easy computation establishes the formula

$$\delta D\psi + D\delta\psi = f^\#(\psi \mid A) - \psi$$

For every $V \in \mathcal{V}$, $(V \cap N) \cup f(V \cap N) \subset W$ for some $W \in \mathcal{W}$ (by lemma 1b), and because $\delta\varphi$ vanishes on $\mathcal{W}^{q+2}$, $\delta D(\varphi \mid N) = f^\#(\varphi \mid A) - \varphi \mid N$ on $\mathcal{V}^{q+1} \cap N^{q+1}$. Therefore the cohomology class $\{\varphi\} \in \bar{H}^q(N')$ maps to zero in $\bar{H}^q(N)$. This suffices to show that $\lim_\rightarrow \{\bar{H}^q(N)\} \to \bar{H}^q(A)$ is a monomorphism, and so A is a taut subspace of X. ■

3 COROLLARY *Let $X \supset A \supset B$, where X is a paracompact Hausdorff space and A and B are closed subspaces of X. Then, relative to the Alexander cohomology theory, (A,B) is a taut pair in X.*

PROOF This is an immediate consequence of theorem 2 and lemma 6.1.9. ■

4 EXAMPLE Let X be the subspace of $\mathbf{R}^2 \subset S^2$ defined in example 2.4.8. The space $\tilde{X}$ obtained by retopologizing X by the topology generated by the path components of open sets in X is a half-open interval. Since X has the same singular homology as $\tilde{X}$, $H^1(X;\mathbf{Z}) = 0$. Since $S^2 - X$ has two components, it follows from the Alexander duality theorem that $\lim_\rightarrow \{H^1(U;\mathbf{Z})\} = \mathbf{Z}$ as U varies over neighborhoods of X. Therefore $\lim_\rightarrow \{H^1(U;\mathbf{Z})\} \to H^1(X;\mathbf{Z})$ is not a monomorphism, and so X is not a taut subspace of $\mathbf{R}^2$ with respect to singular cohomology. Since X is closed in $\mathbf{R}^2$, it is taut with respect to Alexander cohomology.

Note that the above example is one in which $\lim_\rightarrow \{H^1(U;\mathbf{Z})\} \to H^1(X;\mathbf{Z})$ is not a monomorphism, whereas in example 6.1.8 a subspace $A \subset \mathbf{R}^2$ was given such that $\lim_\rightarrow \{H^0(U;\mathbf{Z})\} \to H^0(A;\mathbf{Z})$ was not an epimorphism.

The tautness property 3 implies that the Alexander cohomology theory satisfies the following *strong excision property*.

5 THEOREM *Let (X,A) and (Y,B) be pairs, with X and Y paracompact Hausdorff and A and B closed. Let f: $(X,A) \to (Y,B)$ be a closed continuous map such that f induces a one-to-one map of $X - A$ onto $Y - B$. Then, for all q and all G*

$$f^*: \bar{H}^q(Y,B;\, G) \approx \bar{H}^q(X,A;\, G)$$

PROOF Because f is closed, continuous, and one-to-one from $X - A$ onto $Y - B$, it follows that f is a homeomorphism of $X - A$ onto $Y - B$. Let $\{U_\alpha\}$ be the family of open neighborhoods of B in Y and let $V_\alpha = f^{-1}(U_\alpha)$. Then V_α is an open neighborhood of A in X, and because f is a closed map, the collection $\{V_\alpha\}$ is cofinal in the family of all neighborhoods of A in X. We have a commutative diagram

$$\begin{array}{ccccc}
\bar{H}^q(Y,B) & \leftarrow & \lim_{\to} \{\bar{H}^q(Y,U_\alpha)\} & \to & \lim_{\to} \{\bar{H}^q(Y - B,\, U_\alpha - B)\} \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f_1^*}\downarrow & & \downarrow{\scriptstyle f_2^*} \\
\bar{H}^q(X,A) & \leftarrow & \lim_{\to} \{\bar{H}^q(X,V_\alpha)\} & \to & \lim_{\to} \{\bar{H}^q(X - A,\, V_\alpha - A)\}
\end{array}$$

in which the vertical maps are induced by f and the horizontal maps are induced by inclusions. By corollary 3 and lemma 6.4.4, the horizontal maps are isomorphisms. Because $f \mid X - A$ is a homeomorphism of $X - A$ onto $Y - B$, it follows that for each α, $f \mid (X - A,\, V_\alpha - A)$ is a homeomorphism of $(X - A,\, V_\alpha - A)$ onto $(Y - B,\, U_\alpha - B)$. Therefore f_2^* is an isomorphism, and by commutativity of the diagram, f^* is also an isomorphism. ■

The following *weak continuity property* of the Alexander cohomology theory is another consequence of its tautness properties.

6 THEOREM *Let $\{(X_\alpha,A_\alpha)\}_\alpha$ be a family of compact Hausdorff pairs in some space, directed downward by inclusion, and let $(X,A) = (\bigcap X_\alpha, \bigcap A_\alpha)$. The inclusion maps i_α: $(X,A) \subset (X_\alpha,A_\alpha)$ induce an isomorphism*

$$\{i_\alpha^*\}: \lim_{\to} \bar{H}^q(X_\alpha,A_\alpha;\, M) \approx \bar{H}^q(X,A;\, M)$$

PROOF If F is a closed subset of X_β for some β, the collection $\{X_\alpha \cap F\}_\alpha$ consists of compact sets directed downward by inclusion, and $X \cap F = \bigcap (X_\alpha \cap F)$. It follows that if $X \cap F = \varnothing$, there is some α such that $X_\alpha \cap F = \varnothing$. Therefore, if U is any neighborhood of X in X_β, there exists α such that $X_\alpha \subset U$. Similarly, if (U,V) is any neighborhood of (X,A) in X_β, there is α such that $(X_\alpha,A_\alpha) \subset (U,V)$.

To show that $\{i_\alpha^*\}$ is an epimorphism, let $u \in \bar{H}^q(X,A)$ be arbitrary. For any β, (X,A) is a taut pair in X_β, by corollary 3. Therefore there is a neighborhood (U,V) of (X,A) in X_β and an element $v \in \bar{H}^q(U,V)$ such that $v \mid (X,A) = u$. Let α be such that $(X_\alpha,A_\alpha) \subset (U,V)$ and $v_\alpha = v \mid (X_\alpha,A_\alpha)$. Then $v_\alpha \in \bar{H}^q(X_\alpha,A_\alpha)$ and $i_\alpha^* v_\alpha = u$, which proves that $\{i_\alpha^*\}$ is an epimorphism.

To prove that $\{i_\alpha^*\}$ is a monomorphism, let $u \in \bar{H}^q(X_\beta,A_\beta)$ be such that $i_\beta^* u = 0$. By corollary 3, (X,A) is a taut pair in X_β. Therefore there is a neighborhood (U,V) of (X,A) in X_β such that $u \mid (U,\, V \cap A_\beta) = 0$. Choose α

so that $(X_\alpha,A_\alpha) \subset (U, V \cap A_\beta)$. Then $u \mid (X_\alpha,A_\alpha) = 0$, and $\{i_\alpha^*\}$ is an isomorphism. ■

The *continuity property* involves an assertion analogous to that of theorem 6 for an arbitrary inverse system $\{(X_\alpha,A_\alpha)\}$ of compact Hausdorff pairs, where $(X,A) = \lim_{\leftarrow} \{(X_\alpha,A_\alpha)\}$. It is not hard to prove that the continuity property is equivalent to the weak continuity property.[1] A cohomology theory having the weak continuity property is called *weakly continuous*. Such theories are characterized on the category of compact Hausdorff spaces in view of the following result.

7 LEMMA *Any compact Hausdorff pair can be imbedded in a space in which it is the intersection of a family of pairs directed downward by inclusion, each pair of the family being a compact Hausdorff space of the same homotopy type as a compact polyhedral pair.*

PROOF It is a standard fact that any compact Hausdorff space can be imbedded in a cube I^J; hence we assume (X,A) imbedded in I^J. For each finite subset $\alpha \subset J$ let $p_\alpha\colon I^J \to I^\alpha$ be the projection map and let (U,V) be a compact polyhedral neighborhood of $(p_\alpha(X),p_\alpha(A))$ in I^α. It can be verified that the collection of pairs $\{(p_\alpha^{-1}(U),p_\alpha^{-1}(V))\}$ corresponding to all finite $\alpha \subset J$ and compact polyhedral neighborhoods of $(p_\alpha(X),p_\alpha(A))$ in I^α is directed downward by inclusion and has (X,A) as intersection. Furthermore, $(p_\alpha^{-1}(U),p_\alpha^{-1}(V))$ is a compact pair in I^J homeomorphic to $(U,V) \times I^{J-\alpha}$, and the projection map

$$p_\alpha\colon (p_\alpha^{-1}(U),p_\alpha^{-1}(V)) \to (U,V)$$

is a homotopy equivalence. Therefore the family $\{(p_\alpha^{-1}(U),p_\alpha^{-1}(V))\}$ has the desired properties. ■

This yields the following extension of the uniqueness theorem for weakly continuous cohomology theories.

8 THEOREM *Given two weakly continuous cohomology theories, any homomorphism between them which is an isomorphism for some one-point space is an isomorphism for all compact Hausdorff pairs.* ■

We now describe the Alexander cohomology with compact supports. This is a cohomology theory on a suitable category of topological pairs and maps, and we shall discuss the category first.

A subset A of a topological space X is said to be *bounded* if $\bar{A}$ is compact. A subset $B \subset X$ is said to be *cobounded* if $X - B$ is bounded. A function f from a space X to a space Y is said to be *proper* if it is continuous and if for every bounded set A of Y, $f^{-1}(A)$ is a bounded set of X (or, equivalently, for every cobounded set B of Y, $f^{-1}(B)$ is a cobounded set of X). Clearly, the composite of proper maps is proper, and there is a category of topological spaces and proper maps. There is also a category of topological pairs and

[1] See S. Eilenberg and N. E. Steenrod, "Foundations of Algebraic Topology," Princeton University Press, Princeton, N.J., 1952, or exercise 6.C.2 at the end of this chapter.

proper maps, a proper map from (X,A) to (Y,B) being a proper map from X to Y which maps A to B. This is the category on which the Alexander cohomology theory with compact supports will be defined.

Given a topological pair (X,A), let $C_c{}^q(X,A;\, G)$ be the submodule of $C^q(X,A;\, G)$ consisting of all $\varphi \in C^q(X,A;\, G)$ such that φ is locally zero on some cobounded subset of X. If φ is locally zero on B, so is $\delta\varphi$, and therefore there is a cochain complex $C_c^*(X,A;\, G) = \{C_c{}^q(X,A;\, G), \delta\}$ which is a subcomplex of $C^*(X,A;\, G)$. Clearly, $C_0^*(X;G) \subset C_c^*(X,A;\, G)$, and we define

$$\bar{C}_c^*(X,A;\, G) = C_c^*(X,A;\, G)/C_0^*(X;G)$$

The *Alexander cohomology of* (X,A) *with compact supports*, denoted by $\bar{H}_c^*(X,A;\, G)$, is the cohomology module of $\bar{C}_c^*(X,A;\, G)$. If f: $(X,A) \to (Y,B)$ is a proper map, $f\#$ maps $C_c^*(Y,B;\, G)$ to $C_c^*(X,A;\, G)$ and induces a homomorphism

$$f^*: \bar{H}_c^*(Y,B;\, G) \to \bar{H}_c^*(X,A;\, G)$$

The Alexander cohomology with compact supports satisfies suitable modifications of all the axioms of cohomology theory.

The homotopy axiom holds for proper homotopies, a proper homotopy being a proper map $(X,A) \times I \to (Y,B)$. In general, an inclusion map $(X',A') \subset (X,A)$ is not a proper map. It is a proper map, however, if X' is closed in X. Because of this, the coboundary homomorphism

$$\delta^*: \bar{H}_c{}^q(A;G) \to \bar{H}_c{}^{q+1}(X,A;\, G)$$

is defined only when A is a closed subset of X. When A is a closed subset of X, there are proper inclusion maps i: $A \subset X$ and j: $X \subset (X,A)$ and there is a short exact sequence of cochain complexes (for any coefficient module G)

$$0 \to \bar{C}_c^*(X,A) \xrightarrow{j^\#} \bar{C}_c^*(X) \xrightarrow{i^\#} \bar{C}_c^*(A) \to 0$$

The connecting homomorphism of this short exact sequence is a natural transformation from $\bar{H}_c^*(A)$ to $\bar{H}_c^*(X,A)$, of degree 1 on the category of pairs (X,A), with A closed in X and proper maps between such pairs. The exactness axiom then holds for pairs (X,A) with A closed in X.

The excision axiom holds for proper excisions, a proper excision map being an inclusion map j: $(X - U, A - U) \subset (X,A)$ such that U is an open subset of X with $\bar{U} \subset \operatorname{int} A$, in which case it can be shown (analogous to the proof of lemma 6.4.4) that

$$j\#: \bar{C}_c^*(X,A) \approx \bar{C}_c^*(X - U, A - U)$$

The dimension axiom is obviously satisfied.

We now consider relations between the Alexander cohomology with compact supports and the Alexander cohomology theory previously defined. The following is one case in which they agree.

9 LEMMA *If A is a cobounded subset of X, then*

$$\bar{H}_c^*(X,A;\, G) = \bar{H}^*(X,A;\, G)$$

PROOF Because A is cobounded in X,

$$C_c^*(X,A) = C^*(X,A)$$

and so $\bar{C}_c^*(X,A) = \bar{C}^*(X,A)$. ∎

10 LEMMA *Let B be a closed subset of a Hausdorff space A. Then a subset U of $A - B$ is cobounded in $A - B$ if and only if $U \cup B$ is a neighborhood of B cobounded in A.*

PROOF If U' is a neighborhood of B in A, then the closure of $A - U'$ in A equals the closure of $(A - B) - (U' - B)$ in $A - B$. Hence one is compact if and only if the other is. Therefore the result will follow once we have verified that if U is a cobounded subset of $A - B$, then $U \cup B$ is a neighborhood of B in A. However, if C is the compact set which equals the closure of $(A - B) - U$ in $A - B$, then C is closed in A (because A is Hausdorff). Therefore $A - C$ is an open subset of A containing B. Since $(A - B) - C \subset U$, it follows that $(A - C) \subset U \cup B$, and $U \cup B$ is a neighborhood of B in A. ∎

Let B be a closed subset of a normal space A. If U is a neighborhood of B in A which is a cobounded subset of A, then $\bar{C}^*(A,U) \subset \bar{C}_c^*(A,B)$. Therefore $\lim_{\rightarrow} \{\bar{C}^*(A,U)\} = \bigcup \bar{C}^*(A,U)$ is imbedded as a subcomplex of $\bar{C}_c^*(A,B)$. By the excision property 6.4.4,

$$\bigcup \bar{C}^*(A,U) \approx \bigcup \bar{C}^*(A - B, U - B)$$

As U varies over cobounded neighborhoods of B in A, it follows from lemma 10 that $U - B$ varies over cobounded subsets of $A - B$. Therefore

$$\bigcup \bar{C}^*(A - B, U - B) = \bar{C}_c^*(A - B)$$

and we have defined a functorial imbedding

$$j\colon \bar{C}_c^*(A - B) \subset \bar{C}_c^*(A,B)$$

such that $j(\bar{C}_c^*(A - B)) = \lim_{\rightarrow} \{\bar{C}^*(A,U)\}$, where U varies over cobounded neighborhoods of B in A. Hence j induces an isomorphism of cohomology if and only if

$$\lim_{\rightarrow} \{\bar{H}^*(A,U)\} \approx \bar{H}_c^*(A,B)$$

We shall now consider cases in which j induces an isomorphism of cohomology.

11 LEMMA *If A is a compact Hausdorff space and B is closed in A, for all q and all G there is an isomorphism*

$$\bar{H}_c{}^q(A - B; G) \approx \bar{H}^q(A,B; G)$$

PROOF By lemma 9 and the above remarks, it suffices to prove that as U varies over neighborhoods of B in A (any such neighborhood being cobounded because A is compact), there is an isomorphism

$$\lim_{\rightarrow} \{\bar{H}^q(A,U; G)\} \approx \bar{H}^q(A,B; G)$$

Since A is paracompact, this is a consequence of the tautness property 3 of Alexander cohomology. ■

This result allows the following interpretation of the cohomology with compact supports of a locally compact space.

12 COROLLARY *If X is a locally compact Hausdorff space and X^+ is the one-point compactification of X, there is an isomorphism*

$$\bar{H}_c{}^q(X;G) \approx \tilde{\bar{H}}^q(X^+;G)$$

PROOF By lemma 11, $\bar{H}_c{}^q(X;G) \approx \bar{H}^q(X^+, X^+ - X; G)$ and because $\tilde{\bar{H}}^*(X^+ - X; G) = 0$, there is an isomorphism

$$\bar{H}^q(X^+, X^+ - X; G) \approx \tilde{\bar{H}}^q(X^+;G) \quad ■$$

13 EXAMPLE It follows from corollary 12 that

$$\bar{H}_c{}^q(\mathbf{R}^n;G) \approx \begin{cases} 0 & q \neq n \\ G & q = n \end{cases}$$

because $(\mathbf{R}^n)^+$ is homeomorphic to S^n. Hence, if $n \neq m$, $\mathbf{R}^n$ and $\mathbf{R}^m$ are not of the same proper homotopy type.

14 EXAMPLE Regarding $\mathbf{R}^1$ as a linear subspace of $\mathbf{R}^2$, then

$$\bar{H}_c{}^q(\mathbf{R}^2,\mathbf{R}^1; G) \approx \begin{cases} 0 & q \neq 2 \\ G \oplus G & q = 2 \end{cases}$$

15 THEOREM *Let B be a closed subset of a locally compact Hausdorff space A. For all q and all G there is an isomorphism*

$$\lim_{\rightarrow} \{\bar{H}^q(A,U; G)\} \approx \bar{H}_c{}^q(A,B; G)$$

where U varies over cobounded neighborhoods of B in A.

PROOF If A is compact, this follows from lemmas 9 and 11. If A is not compact, let A^+ be the one-point compactification of A. Set $p^+ = A^+ - A$ and $B^+ = B \cup p^+ \subset A^+$. Then B^+ is closed in the compact space A^+. There is a commutative diagram of chain maps

$$\begin{array}{ccccccccc} & & \bar{C}_c^*(A - B) & \rightarrow & \bar{C}_c^*(A) & \rightarrow & \bar{C}_c^*(B) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \bar{C}^*(A^+,B^+) & \rightarrow & \bar{C}^*(A^+,p^+) & \rightarrow & \bar{C}^*(B^+,p^+) & \rightarrow & 0 \end{array}$$

and, by corollary 12 and lemma 11, each vertical map induces an isomorphism on cohomology. Since the bottom row is exact and $\bar{C}_c^*(A - B) \subset \bar{C}_c^*(A)$, it follows that $\bar{C}_c^*(A)/\bar{C}_c^*(A - B) \rightarrow \bar{C}_c^*(B)$ induces isomorphisms of cohomology. Since there is a short exact sequence of cochain complexes

$$0 \rightarrow \bar{C}_c^*(A,B)/\bar{C}_c^*(A - B) \rightarrow \bar{C}_c^*(A)/\bar{C}_c^*(A - B) \rightarrow \bar{C}_c^*(B) \rightarrow 0$$

it follows that $\bar{C}_c^*(A,B)/\bar{C}_c^*(A - B)$ has trivial cohomology. Therefore

$\bar{H}^*_c(A - B) \approx \bar{H}^*_c(A,B)$, and this is equivalent to the statement of the theorem. ∎

The last result is a form of tautness for Alexander cohomology with compact supports. This and the five lemma easily imply the next result.

16 THEOREM *Let (A,B) be a pair of closed subsets of a locally compact Hausdorff space X. For all q and all G there is an isomorphism*

$$\lim_{\rightarrow} \{\bar{H}^q(U,V;\, G)\} \approx \bar{H}_c{}^q(A,B;\, G)$$

where (U,V) varies over neighborhoods of (A,B) in X, both U and V being cobounded subsets of X. ∎

In a similar fashion, we may consider the singular cohomology with compact supports. A singular cochain $c^* \in \operatorname{Hom}(\Delta_q(X)/\Delta_q(A),G)$ is said to have *compact support* if there is some cobounded set $U \subset X$ such that for every $x \in U$ there is a neighborhood V of x such that $c^*(\sigma) = 0$ for every singular q-simplex σ in V. The singular cochains with compact support form a subcomplex of the singular cochain complex, whose cohomology module is denoted by $H^*_c(X,A;\, G)$.

7 PRESHEAVES

In this section the Čech construction will be introduced. Because of the ultimate applications, we define the Čech cohomology of a space not merely for coefficients in a module, but, more generally, for coefficient modules which may vary from one point of X to another. This leads to the concepts of presheaf and sheaf. We shall introduce these and give the definition of the Čech cohomology of a space with coefficients in a presheaf. Applications will be given in the next two sections.

A *presheaf* Γ of R modules on a topological space X is a contravariant functor from the category of open subsets U of X and inclusion maps $U \subset V$ to the category of R modules such that $\Gamma(\varnothing) = 0$. Thus Γ assigns to every open subset $U \subset X$ an R module $\Gamma(U)$ and to every inclusion map $U \subset V$ a homomorphism $\rho_{UV}\colon \Gamma(V) \to \Gamma(U)$, called the *restriction map,* such that

$$\rho_{UU} = 1_{\Gamma(U)}$$
$$\rho_{UW} = \rho_{UV} \circ \rho_{VW}\colon \Gamma(W) \to \Gamma(U) \qquad U \subset V \subset W$$

Given $\gamma \in \Gamma(V)$ and $U \subset V$, we use $\gamma \mid U$ to denote the image $\rho_{UV}(\gamma) \in \Gamma(U)$.

In a similar manner, we define presheaves on X with values in any category. We are interested primarily in the case of a presheaf of modules or of cochain complexes. Following are some examples.

1 Given an R module G, the *constant presheaf* G on X assigns to every nonempty open $U \subset X$, the module G (and to $\varnothing$ the trivial module).

2 Given a subset $A \subset X$, the *relative Alexander presheaf of* (X,A) *with* *coefficients* G, denoted by $C^*(\cdot, \cdot \cap A; G)$, assigns to an open $U \subset X$ the cochain complex $C^*(U, U \cap A; G)$.

3 The *relative singular presheaf of* (X,A) *with coefficients* G, denoted by $\Delta^*(\cdot, \cdot \cap A; G)$, assigns to an open $U \subset X$ the cochain complex $\Delta^*(U, U \cap A; G)$ equal to the subcomplex of Hom $(\Delta_*(U), G)$ of cochains locally zero on $U \cap A$ (i.e. cochains that are zero on $\Delta_*(\mathfrak{U})$ for some open covering $\mathfrak{U}$ of $U \cap A$).

Given two presheaves Γ and Γ' on X taking values in the same category, a *homomorphism* α: $\Gamma \to \Gamma'$ is defined to be a natural transformation from Γ to Γ'. It is then clear that there is a category of presheaves on X with values in any fixed category and homomorphisms between them. In particular, there is a category of presheaves of modules and a category of presheaves of cochain complexes. If α: $\Gamma \to \Gamma'$ is a homomorphism of presheaves of modules (or cochain complexes), it is clear how to define ker α, im α, and coker α so as to be presheaves of modules (or cochain complexes) on X. Therefore it is meaningful to consider exact sequences of presheaves of modules (or cochain complexes) on X.

If Γ and Γ' are presheaves of modules (or cochain complexes) on X, their *tensor product* $\Gamma \otimes \Gamma'$ is the presheaf of modules (or cochain complexes) on X such that for open $U \subset X$

$$(\Gamma \otimes \Gamma')(U) = \Gamma(U) \otimes \Gamma'(U)$$

Consider two examples.

4 There is a homomorphism

$$\tau\colon C^*(\cdot, \cdot \cap A; G) \to \Delta^*(\cdot, \cdot \cap A; G)$$

such that if $\varphi \in C^q(U, U \cap A; G)$ and σ: $\Delta^q \to U$, then $\tau(\varphi)(\sigma) = \varphi(\sigma(p_0), \ldots, \sigma(p_q))$, where $p_0, \ldots, p_q$ are the vertices of Δ^q.

5 There is a homomorphism

$$\tau\colon C^*(\cdot, \cdot \cap A; R) \otimes G \to C^*(\cdot, \cdot \cap A; G)$$

such that if $\varphi \in C^q(U, U \cap A; R)$ and $g \in G$, then

$$\tau(\varphi \otimes g)(x_0, \ldots, x_q) = \varphi(x_0, \ldots, x_q)g \qquad x_i \in U$$

Similar to the concept of presheaf on X with values in a category is the concept of sheaf on X with values in a category. We are interested only in sheaves of modules, and for this case the following formulation will do.

Let Γ be a presheaf of modules on X. If $\mathfrak{U} = \{U\}$ is a collection of open subsets of X, a *compatible* $\mathfrak{U}$ *family of* Γ is an indexed family $\{\gamma_U \in \Gamma(U)\}_{U \in \mathfrak{U}}$ such that

$$\gamma_U \mid U \cap U' = \gamma_{U'} \mid U \cap U' \qquad U, U' \in \mathfrak{U}$$

The presheaf Γ is said to be a *sheaf* if both the following conditions hold:

(*a*) Given a collection $\mathcal{U}$ of open subsets of X with $V = \bigcup_{U \in \mathcal{U}} U$ and given $\gamma \in \Gamma(V)$ such that $\gamma \mid U = 0$ for all $U \in \mathcal{U}$, then $\gamma = 0$.

(*b*) Given a collection $\mathcal{U}$ of open subsets of X with $V = \bigcup_{U \in \mathcal{U}} U$ and given a compatible $\mathcal{U}$ family $\{\gamma_U\}_{U \in \mathcal{U}}$, there is an element $\gamma \in \Gamma(V)$ such that $\gamma \mid U = \gamma_U$ for all $U \in \mathcal{U}$.

It follows from (*a*) that the element γ in (*b*) is unique.

We now associate to every presheaf Γ of modules another presheaf $\hat{\Gamma}$, called its completion, whose elements are compatible families of Γ. Given a collection of open sets $\mathcal{U} = \{U\}$, let $\Gamma(\mathcal{U})$ be the module of compatible $\mathcal{U}$ families of Γ. If $\mathcal{V}$ is another collection of open sets which refines $\mathcal{U}$, there is a homomorphism $\Gamma(\mathcal{U}) \to \Gamma(\mathcal{V})$ which assigns to a compatible $\mathcal{U}$ family $\{\gamma_U\}$ the compatible $\mathcal{V}$ family $\{\gamma'_V\}$ such that if $V \in \mathcal{V}$ is contained in $U \in \mathcal{U}$, then $\gamma'_V = \gamma_U \mid V$ (γ'_V is uniquely defined by this condition because of the compatibility of $\{\gamma_U\}$). As $\mathcal{U}$ varies over the family of open coverings of a fixed open set $W \subset X$, the collection $\{\Gamma(\mathcal{U})\}$ is a direct system of modules, and we define

$$\hat{\Gamma}(W) = \lim_{\to} \{\Gamma(\mathcal{U})\}$$

If $W' \subset W$ and $\mathcal{U}$ is an open covering of W, then $\mathcal{U}' = \{U \cap W' \mid U \in \mathcal{U}\}$ is an open covering of W' which refines $\mathcal{U}$. Hence there is a homomorphism $\Gamma(\mathcal{U}) \to \Gamma(\mathcal{U}')$ which defines (by passage to the limit) a homomorphism $\hat{\Gamma}(W) \to \hat{\Gamma}(W')$. A trivial verification shows that $\hat{\Gamma}$ is a presheaf [if $\mathcal{U} = \{\varnothing\}$, then trivially $\Gamma(\mathcal{U}) = 0$, and so $\hat{\Gamma}(\varnothing) = 0$]. There is a natural homomorphism $\alpha: \Gamma \to \hat{\Gamma}$ such that α assigns to $\gamma \in \Gamma(V)$ the element of $\hat{\Gamma}(V)$ represented by the compatible $\mathcal{V}$ family $\{\gamma\}$, where $\mathcal{V}$ consists solely of V. The presheaf $\hat{\Gamma}$ is called the *completion* of Γ. It depends only on the values $\Gamma(U)$ for small open sets $U \subset X$.

6 LEMMA *A presheaf Γ is a sheaf if and only if*

$$\alpha: \Gamma \approx \hat{\Gamma}$$

PROOF In fact, condition (*a*) above is satisfied if and only if α is a monomorphism. If condition (*b*) is satisfied, α is an epimorphism. If α is an isomorphism, then (*b*) is satisfied. ∎

7 EXAMPLE The constant presheaf G defined by a module G is not generally a sheaf [if U is a disconnected open set, $G(U) \not\approx \hat{G}(U)$].

8 EXAMPLE If C^* is the relative Alexander presheaf of (X,A) (with some coefficient module G), the kernel of $\alpha: C^* \to \hat{C}^*$ is C_0^* (the locally zero functions). To show that α satisfies condition (*b*) (and hence induces an isomorphism $\bar{C}^* \approx \hat{C}^*$), let $\varphi' \in \hat{C}^q(V, V \cap A)$ and assume φ' represented by a compatible $\mathcal{U}$ family $\{\varphi_U\}_{U \in \mathcal{U}}$, where $\mathcal{U}$ is an open covering of V. Then $\varphi_U: U^{q+1} \to G$ for $U \in \mathcal{U}$ is locally zero on $U \cap A$ and

$$\varphi_U \mid (U \cap U')^{q+1} = \varphi_{U'} \mid (U \cap U')^{q+1} \qquad U, U' \in \mathcal{U}$$

Therefore there is a function $\varphi\colon V^{q+1} \to G$ such that $\varphi \mid U^{q+1} = \varphi_U$ for $U \in \mathcal{U}$ and $\varphi(x_0, \ldots, x_q) = 0$ if $x_0, \ldots, x_q$ do not all lie in some element of $\mathcal{U}$. Then φ is locally zero on A, whence $\varphi \in C^q(V, V \cap A)$ and $\alpha(\varphi) = \varphi'$.

This example shows that, in general, $H^*(C^*) \neq H^*(\hat{C}^*)$, so it is not generally true that a presheaf of cochain complexes and its completion have isomorphic cohomology.

9 EXAMPLE If Δ^* is the relative singular presheaf of (X,A) (with some coefficient module G), the kernel of $\alpha\colon \Delta^* \to \hat{\Delta}^*$ is the subcomplex of locally zero cochains [that is, $c^* \in \operatorname{Hom}(\Delta_q(V),G)$ is in the kernel of α if and only if there is some open covering $\mathcal{U}$ of V such that c^* is zero on $\Delta_q(\mathcal{U}) \subset \Delta_q(V)$]. Also α satisfies condition (b) (as can be shown by an argument similar to that of example 8). If $\mathcal{U}$ is an open covering of X, it is clear that $\Delta^*(\mathcal{U}) = \bigcup \operatorname{Hom}(\Delta_*(\mathcal{U})/\Delta_*(\mathcal{U}'), G)$ where the union is over all open coverings $\mathcal{U}'$ of A that refine $\mathcal{U} \cap A$. As $\mathcal{U}$ and $\mathcal{U}'$ vary over open coverings, respectively, of X and A such that $\mathcal{U}'$ refines $\mathcal{U} \cap A$, there is an inverse system of chain complexes $\{\Delta_*(\mathcal{U})/\Delta_*(\mathcal{U}')\}$ and a direct system of cochain complexes

$$\{\operatorname{Hom}(\Delta_*(\mathcal{U})/\Delta_*(\mathcal{U}'), G)\}$$

Therefore there is an isomorphism

$$\varinjlim \{\operatorname{Hom}(\Delta_*(\mathcal{U})/\Delta_*(\mathcal{U}'), G)\} \approx \hat{\Delta}^*(\cdot\,, \cdot \cap A; G)(X)$$

It follows from theorem 4.4.14 that

$$\operatorname{Hom}(\Delta_*(X)/\Delta_*(A), G) \to \varinjlim \{\operatorname{Hom}(\Delta_*(\mathcal{U})/\Delta_*(\mathcal{U}'), G)\}$$

induces isomorphisms of the cohomology modules. Therefore α induces an isomorphism

$$H^*(\Delta^*(\cdot\,, \cdot \cap A; G)(X)) \approx H^*(\hat{\Delta}^*(\cdot\,, \cdot \cap A; G)(X))$$

10 EXAMPLE Let ξ be an n-sphere bundle with base space B and let R be fixed. A presheaf Γ on B is defined by $\Gamma(V) = H^{n+1}(p_\xi^{-1}(V), p_\xi^{-1}(V) \cap \dot{E}_\xi; R)$ for an open $V \subset B$. Γ is called the *orientation presheaf of ξ over R*. It can be verified that if B is connected, ξ is orientable over R if and only if $\hat{\Gamma}(B) \neq 0$.

11 EXAMPLE Let X be an n-manifold with boundary $\dot{X}$ and let R be fixed. Define a presheaf Γ on $X - \dot{X}$ such that $\Gamma(V) = H_n(X, X - V; R)$ for open $V \subset X - \dot{X}$. Γ is called the *fundamental presheaf of X over R*. It can be verified (using lemma 6.3.2) that $\hat{\Gamma}(X) \approx H_n^c(X,\dot{X}; R)$. By theorem 6.3.5, it follows that if X is connected, it is orientable over R if and only if $\hat{\Gamma}(X) \neq 0$.

There are cohomology modules of X with coefficients in sheaves,[1] and cohomology modules with coefficients in presheaves. For paracompact spaces

[1] See R. Godement, "Théorie des faisceaux," Hermann et Cie, Paris, 1958.

these theories are equivalent. We now define the Čech cohomology with coefficients in a presheaf of modules.

Let Γ be a presheaf of modules on a space X and let $\mathcal{U}$ be an open covering of X. For $q \geq 0$ define $C^q(\mathcal{U};\Gamma)$ to be the module of functions ψ which assign to an ordered $(q+1)$-tuple $U_0, U_1, \ldots, U_q$ of elements of $\mathcal{U}$ an element $\psi(U_0, \ldots, U_q) \in \Gamma(U_0 \cap \cdots \cap U_q)$. A coboundary operator

$$\delta\colon C^q(\mathcal{U};\Gamma) \to C^{q+1}(\mathcal{U};\Gamma)$$

is defined by

$$(\delta\psi)(U_0, \ldots, U_{q+1}) = \sum_{0 \leq i \leq q+1} (-1)^i \psi(U_0, \ldots, \hat{U}_i, \ldots, U_{q+1}) \mid (U_0 \cap \cdots \cap U_{q+1})$$

Then $\delta\delta = 0$ and $C^*(\mathcal{U};\Gamma) = \{C^q(\mathcal{U};\Gamma),\delta\}$ is a cochain complex. Its cohomology module is denoted by $H^*(\mathcal{U};\Gamma)$.

12 EXAMPLE It is an immediate consequence of the definition that $H^0(\mathcal{U};\Gamma) = \Gamma(\mathcal{U})$ (the module of compatible $\mathcal{U}$ families).

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$ and let $\lambda\colon \mathcal{V} \to \mathcal{U}$ be a function such that $V \subset \lambda(V)$ for all $V \in \mathcal{V}$. There is a cochain map $\lambda^*\colon C^*(\mathcal{U};\Gamma) \to C^*(\mathcal{V};\Gamma)$ defined by

$$(\lambda^*\psi)(V_0, \ldots, V_q) = \psi(\lambda(V_0), \ldots, \lambda(V_q)) \mid (V_0 \cap \cdots \cap V_q)$$

If $\mu\colon \mathcal{V} \to \mathcal{U}$ is another function such that $V \subset \mu(V)$ for all $V \in \mathcal{V}$, a cochain homotopy $D\colon C^q(\mathcal{U};\Gamma) \to C^{q-1}(\mathcal{V};\Gamma)$ from λ^* to μ^* is defined by

$$(D\psi)(V_0, \ldots, V_{q-1}) = \sum_{0 \leq j \leq q-1} (-1)^j \psi(\lambda(V_0), \ldots, \lambda(V_j), \mu(V_j), \ldots, \mu(V_{q-1})) \mid (V_0 \cap \cdots \cap V_{q-1})$$

It follows that there is a well defined homomorphism

$$\lambda^*\colon H^*(\mathcal{U};\Gamma) \to H^*(\mathcal{V};\Gamma)$$

such that $\lambda^*\{\varphi\} = \{\lambda^*\varphi\}$ that is independent of the particular choice of λ. As $\mathcal{U}$ varies over open coverings of X, the collection $\{H^*(\mathcal{U};\Gamma)\}$ is a direct system, and the *Čech cohomology of X with coefficients* Γ is defined by

$$\check{H}^*(X;\Gamma) = \lim_{\rightarrow} \{H^*(\mathcal{U};\Gamma)\}$$

13 EXAMPLE For any presheaf Γ, $\check{H}^0(X;\Gamma) = \hat{\Gamma}(X)$.

14 EXAMPLE The *Čech cohomology of X with coefficients G*, denoted by $\check{H}^*(X;G)$, is defined to be the cohomology of X with coefficients the constant presheaf G.

We now establish some basic properties of the cohomology with coefficients in a presheaf.

15 THEOREM *There is a covariant functor from the category of short exact sequences of presheaves on X to the category of exact sequences which assigns to a short exact sequence $0 \to \Gamma' \to \Gamma \to \Gamma'' \to 0$ of presheaves on X an exact sequence*

$$\cdots \to \check{H}^q(X;\Gamma') \to \check{H}^q(X;\Gamma) \to \check{H}^q(X;\Gamma'') \to \check{H}^{q+1}(X;\Gamma') \to \cdots$$

PROOF For any open covering $\mathfrak{U}$ there is a short exact sequence of cochain complexes

$$0 \to C^*(\mathfrak{U};\Gamma') \to C^*(\mathfrak{U};\Gamma) \to C^*(\mathfrak{U};\Gamma'') \to 0$$

This yields an exact cohomology sequence, and the result follows from this on passing to the direct limit. ∎

Given a short exact sequence of modules

$$0 \to G' \to G \to G'' \to 0$$

the corresponding constant presheaves on X constitute a short exact sequence of presheaves. The corresponding exact cohomology sequence of Čech cohomology modules given by theorem 15 is an analogue for Čech theory of the exact sequence of theorem 5.4.11.

Given a presheaf Γ on X and given a subspace $A \subset X$, define a presheaf Γ_A on X by

$$\Gamma_A(U) = \begin{cases} \Gamma(U) & U \cap A \neq \varnothing \\ 0 & U \cap A = \varnothing \end{cases}$$

Also define a presheaf Γ^A on X by

$$\Gamma^A(U) = \begin{cases} \Gamma(U) & U \cap A = \varnothing \\ 0 & U \cap A \neq \varnothing \end{cases}$$

Then Γ^A is a sub-presheaf of Γ, and there is a short exact sequence of presheaves

$$0 \to \Gamma^A \to \Gamma \to \Gamma_A \to 0$$

The corresponding exact cohomology sequence given by theorem 15 is an exact Čech cohomology sequence of the pair (X,A) with coefficients Γ when we define $\check{H}^q(A;\Gamma) = \check{H}^q(X,\Gamma_A)$ and $\check{H}^q(X,A;\,\Gamma) = \check{H}^q(X;\Gamma^A)$. Thus the exact sequence of theorem 15 gives rise to exact sequences corresponding to a change of coefficients or to a change of space.

A presheaf Γ of modules on X is said to be *locally zero* if, given $\gamma \in \Gamma(V)$, there is an open covering $\mathfrak{U}$ of V such that $\gamma \mid U = 0$ for all $U \in \mathfrak{U}$. This is so if and only if the completion $\hat{\Gamma}$ of Γ is the zero presheaf and is equivalent to the condition that for all $x \in X$, $\lim_{\to} \{\Gamma(U)\} = 0$ as U varies over open neighborhoods of x.

16 THEOREM *If X is a paracompact Hausdorff space and Γ is a locally zero presheaf on X, then $\check{H}^*(X;\Gamma) = 0$.*

PROOF Let $\mathfrak{U}$ be a locally finite open covering of X and φ a q-cochain of

$C^*(\mathscr{U};\Gamma)$. Let $\mathscr{W}$ be a locally finite open star refinement of $\mathscr{U}$. For $x \in X$, because Γ is locally zero, there is an open neighborhood V_x contained in some element of $\mathscr{W}$ such that $x \in U_0 \cap \cdots \cap U_q$ with $U_0, \ldots, U_q \in \mathscr{U}$ implies that $V_x \subset U_0 \cap \cdots \cap U_q$ and $\varphi(U_0, \ldots, U_q) \mid V_x = 0$ (only a finite number of conditions because $\mathscr{U}$ is locally finite). Let $\mathscr{V} = \{V_x\}_{x \in X}$ and define $\lambda: \mathscr{V} \to \mathscr{U}$ so that for each $x \in X$ there is $W_x \in \mathscr{W}$ with $V_x \subset W_x \subset W_x^* \subset \lambda(V_x)$. Then if $V_{x_0} \cap \cdots \cap V_{x_q} \neq 0$, $V_{x_0} \subset W_{x_j}$ for each j so that $V_{x_0} \subset \lambda(V_{x_j})$ for each j. Therefore, $\phi(\lambda(V_{x_0}), \ldots, \lambda(V_{x_q})) \mid V_{x_0} = 0$, so $\lambda^*\phi = 0$ in $C^*(\mathscr{V};\Gamma)$. Therefore, $\check{H}^q(X;\Gamma) = 0$ for all q.

A homomorphism $\alpha: \Gamma \to \Gamma'$ between presheaves on X is called a *local isomorphism* if ker α and coker α are both locally zero. This is equivalent to the condition that for all $x \in X$, α induces an isomorphism

$$\lim_{\to} \{\Gamma(U)\} \approx \lim_{\to} \{\Gamma'(U)\}$$

where U varies over open neighborhoods of x. There are short exact sequences of presheaves

$$0 \to \ker \alpha \to \Gamma \xrightarrow{\alpha'} \operatorname{im} \alpha \to 0$$

$$0 \to \operatorname{im} \alpha \xrightarrow{\alpha''} \Gamma' \to \operatorname{coker} \alpha \to 0$$

with $\alpha = \alpha''\alpha'$. Combining theorems 15 and 16, we obtain the following result.

17 COROLLARY *If $\alpha: \Gamma \to \Gamma'$ is a local isomorphism of presheaves on a paracompact Hausdorff space X, then*

$$\alpha_*: \check{H}^*(X;\Gamma) \approx \check{H}^*(X;\Gamma') \quad \blacksquare$$

18 COROLLARY *If X is a paracompact Hausdorff space, the natural homomorphism $\alpha: \Gamma \to \hat{\Gamma}$ induces isomorphisms*

$$\alpha_*: \check{H}^*(X;\Gamma) \approx \check{H}^*(X;\hat{\Gamma})$$

PROOF It suffices to prove that $\alpha: \Gamma \to \hat{\Gamma}$ is a local isomorphism. Let $\gamma \in (\ker \alpha)(V)$. Then $\gamma \in \Gamma(V)$, and there is an open covering $\mathfrak{U}$ of V such that $\gamma \mid U = 0$ for all $U \in \mathfrak{U}$. Hence ker α is locally zero.

If $\gamma' \in (\operatorname{coker} \alpha)(V)$, there is an open covering $\mathfrak{U}$ of V and a compatible $\mathfrak{U}$ family $\{\gamma_U\}$ which represents γ'. For each $U \in \mathfrak{U}$, $\gamma' \mid U$ is represented by $\gamma_U \in \alpha(\Gamma(U))$. Therefore $\gamma' \mid U = 0$, and coker α is locally zero. ■

8 FINE PRESHEAVES

In this section we shall introduce the concept of fine presheaf and show that the positive dimensional cohomology of a paracompact space with coefficients in a fine presheaf is zero. This leads to uniqueness theorems for cohomology of cochain complexes of fine presheaves on a paracompact space, which we apply to compare the Alexander and Čech cohomology. Further applications will be given in the next section.

A presheaf Γ on X is said to be *fine* if, given any locally finite open covering $\mathcal{U}$ of X, there exists an indexed family $\{e_U\}_{U \in \mathcal{U}}$ of endomorphisms of Γ such that (for every open set V in X)

(*a*) For $\gamma \in \Gamma(V)$, $e_U(\gamma) \mid (V - \bar{U}) = 0$.
(*b*) If V meets only finitely many elements of $\{\bar{U}\}$, then for $\gamma \in \Gamma(V)$, $\gamma = \sum_{U \in \mathcal{U}} e_U(\gamma)$.

Note that the sum in condition (*b*) is finite because, by (*a*), $e_U(\gamma) = 0$ if $\bar{U} \cap V = \varnothing$.

1 EXAMPLE The relative Alexander presheaf of (X,A) of degree q with coefficients G is fine. In fact, if $\mathcal{U}$ is a locally finite open covering of X, for each $x \in X$ choose an element $U_x \in \mathcal{U}$ containing x and for $\varphi \in C^q(V, V \cap A; G)$ define $e_U\varphi \in C^q(V, V \cap A; G)$ by

$$(e_U\varphi)(x_0, \ldots, x_q) = \begin{cases} \varphi(x_0, \ldots, x_q) & U = U_{x_0} \\ 0 & U \neq U_{x_0} \end{cases}$$

If $V' \subset V$, there is a commutative square

$$\begin{array}{ccc} C^q(V, V \cap A; G) & \xrightarrow{e_U} & C^q(V, V \cap A; G) \\ \downarrow & & \downarrow \\ C^q(V', V' \cap A; G) & \xrightarrow{e_U} & C^q(V', V' \cap A; G) \end{array}$$

showing that e_U is an endomorphism of C^q. If

$$(x_0, \ldots, x_q) \in (V^{q+1} - \bar{U}^{q+1}) \subset (V^{q+1} - U^{q+1}),$$

then $U_{x_0} \neq U$ and $(e_U\varphi)(x_0, \ldots, x_q) = 0$. Hence $e_U\varphi \mid V - \bar{U} = 0$, and condition (*a*) is satisfied. To show that (*b*) is also satisfied, observe that, given $x_0, \ldots, x_q$, there is a unique U, namely U_{x_0}, such that $(e_U\varphi)(x_0, \ldots, x_q) \neq 0$. Then

$$(\sum e_U\varphi)(x_0, \ldots, x_q) = (e_{U_{x_0}}\varphi)(x_0, \ldots, x_q) = \varphi(x_0, \ldots, x_q)$$

It should be noted that e_U does not commute with the coboundary operator in $C^*(V, V \cap A; G)$. Therefore e_U is not an endomorphism of the Alexander presheaf $C^*(\cdot, \cdot \cap A; G)$ of cochain complexes.

2 EXAMPLE The relative singular presheaf of (X,A) of degree q with coefficients G is also fine. If $\mathcal{U}$ is a locally finite open covering of X and U_x is chosen so that $x \in U_x \in \mathcal{U}$, then

$$e_U\colon \operatorname{Hom}(\Delta_q(V)/\Delta_q(V \cap A), G) \to \operatorname{Hom}(\Delta_q(V)/\Delta_q(V \cap A), G)$$

is defined by

$$(e_U c^*)(\sigma) = \begin{cases} c^*(\sigma) & U = U_{\sigma(p_0)} \\ 0 & U \neq U_{\sigma(p_0)} \end{cases}$$

Then the family $\{e_U\}_{U \in \mathcal{U}}$ satisfies conditions (*a*) and (*b*) of the definition of fineness [but e_U is not an endomorphism of $\Delta^*(\cdot, \cdot \cap A; G)$ so $\Delta^*(\cdot, \cdot \cap A; G)$ has not been shown to be a fine presheaf of cochain complexes].

Given a presheaf Γ on X and a continuous map $f\colon X \to Y$, there is a presheaf $f_* \Gamma$ on Y defined by $(f_* \Gamma)(V) = \Gamma(f^{-1}V)$ for an open $V \subset Y$. Clearly, f defines a covariant functor from the category of presheaves of any type on X to the category of presheaves of the same type on Y. Some of the nice properties of fine presheaves are made explicit in the following result.

3 THEOREM *Let Γ be a fine presheaf of modules on X.*

(*a*) *For any presheaf Γ' of modules on X, $\Gamma \otimes \Gamma'$ is fine.*
(*b*) *If $f\colon X \to Y$ is continuous, $f_* \Gamma$ is fine on Y.*
(*c*) *$\hat{\Gamma}$ is a fine presheaf on X.*

PROOF For (*a*), observe that if $\mathfrak{U}$ is a locally finite open covering of X and $\{e_U\}_{U \in \mathfrak{U}}$ are the corresponding endomorphisms of Γ, then $\{e_U \otimes 1\}_{U \in \mathfrak{U}}$ is a family of endomorphisms of $\Gamma \otimes \Gamma'$, showing that $\Gamma \otimes \Gamma'$ is fine.

For (*b*), observe that if $\mathfrak{U}$ is a locally finite open covering of Y, then $f^{-1}\mathfrak{U} = \{f^{-1}U \mid U \in \mathfrak{U}\}$ is a locally finite open covering of X. If $\{e_U\}_{U \in \mathfrak{U}}$ is a family of endomorphisms of Γ corresponding to the covering $f^{-1}\mathfrak{U}$, they induce endomorphisms of $f_* \Gamma$, showing that $f_* \Gamma$ is fine.

(*c*) follows easily on observing that any endomorphism of Γ induces an endomorphism of $\hat{\Gamma}$. ∎

Given an open covering $\mathfrak{U}$ of a space X, a *shrinking* of $\mathfrak{U}$ is an open covering $\mathfrak{V}$ of X in one-to-one correspondence with $\mathfrak{U}$ such that if $U \in \mathfrak{U}$ corresponds to $V_U \in \mathfrak{V}$, then $\bar{V}_U \subset U$. Any locally finite open covering of a normal Hausdorff space has shrinkings. Any shrinking of a locally finite open covering is clearly locally finite.

The following theorem is the main result on fine presheaves.

4 THEOREM *If Γ is a fine presheaf on a paracompact Hausdorff space X, then $\check{H}^q(X;\Gamma) = 0$ for $q > 0$.*

PROOF Let $\mathfrak{U} = \{U\}$ be a locally finite open covering of X and let $\mathfrak{U}' = \{U'\}$ be a shrinking of $\mathfrak{U}$. Let $\{e_U\}_{U \in \mathfrak{U}}$ be fineness endomorphisms of Γ corresponding to the covering $\mathfrak{U}'$ (but indexed by the covering $\mathfrak{U}$). Let $\mathfrak{V} = \{V\}$ be an open refinement of $\mathfrak{U}$ covering X such that each $V \in \mathfrak{V}$ meets only a finite number of elements of $\mathfrak{U}$ and for any $U \in \mathfrak{U}$ either $V \subset U$ or $V \subset X - \bar{U}'$. Let $\lambda\colon \mathfrak{V} \to \mathfrak{U}$ be a function such that $V \subset \lambda(V)$ for all $V \in \mathfrak{V}$.

Since each e_U is an endomorphism of Γ, e_U induces a cochain map, denoted by $e_U\colon C^*(\mathfrak{U};\Gamma) \to C^*(\mathfrak{U};\Gamma)$ such that for $\psi \in C^q(\mathfrak{U};\Gamma)$ and $U_0, \ldots, U_q \in \mathfrak{U}$

$$(e_U\psi)(U_0, \ldots, U_q) = e_U(\psi(U_0, \ldots, U_q))$$

Then e_U acts similarly as a cochain map on $C^*(\mathfrak{V};\Gamma)$ and commutes with the cochain map $\lambda^*\colon C^*(\mathfrak{U};\Gamma) \to C^*(\mathfrak{V};\Gamma)$.

Let $q > 0$ and $\psi \in C^q(\mathfrak{U};\Gamma)$ be a cocycle. Define $\psi_U \in C^q(\mathfrak{V};\Gamma)$ by

$\psi_U = e_U(\lambda^* \psi)$. Then ψ_U is a cocycle for each $U \in \mathcal{U}$, and if $V_0, \ldots, V_q \in \mathcal{V}$, then $\psi_U(V_0, \ldots, V_q) = 0$, except for a finite number of $U \in \mathcal{U}$. Therefore $\Sigma\, \psi_U$ exists, and $\Sigma\, \psi_U = \lambda^* \psi$.

Define $\psi'_U \in C^{q-1}(\mathcal{V};\Gamma)$ by

$$\psi'_U(V_0, \ldots ,V_{q-1}) = \begin{cases} e_U(\psi(U,\lambda(V_0), \ldots ,\lambda(V_{q-1})) \mid (V_0 \cap \cdots \cap V_{q-1})) & V_0 \cap \cdots \cap V_{q-1} \subset U \\ 0 & V_0 \cap \cdots \cap V_{q-1} \subset X - \bar{U}' \end{cases}$$

Then $\delta\psi'_U = \psi_U$ for all U, and because $\Sigma\, \psi'_U$ can be formed [for given $V_0, \ldots, V_{q-1}, \psi'_U(V_0, \ldots ,V_{q-1}) = 0$, except for a finite number of $U \in \mathcal{U}$], we see that

$$\lambda^* \psi = \Sigma\, \psi_U = \delta(\Sigma\, \psi'_U)$$

Therefore $\lambda^* \psi$ is a coboundary, and $\check{H}^q(X;\Gamma) = 0$. ■

Our next results are technical lemmas about cochain complexes of presheaves. If Γ^* is a cochain complex of presheaves of modules on X, we use Z^q and B^{q+1} to denote the kernel and image, respectively, of $\delta\colon \Gamma^q \to \Gamma^{q+1}$ and H^q to denote Z^q/B^q, all of these being presheaves of modules on X. (Note that a fine presheaf of cochain complexes is a cochain complex of fine presheaves, but the converse is not generally true.)

5 LEMMA *Let Γ^* be a cochain complex of presheaves of modules on X. For every q there is an exact sequence, functorial in Γ^*,*

$$0 \to \ker\,(\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})) \to \check{H}^0(X;Z^q) \to H^q(\hat{\Gamma}^*(X)) \to 0$$

PROOF By example 6.7.13, $\hat{\Gamma}^q(X) = \check{H}^0(X;\Gamma^q)$. From the short exact sequence of presheaves

$$0 \to Z^q \to \Gamma^q \to B^{q+1} \to 0$$

there follows, by theorem 6.7.15, an exact sequence

$$0 \to \check{H}^0(X;Z^q) \to \check{H}^0(X;\Gamma^q) \to \check{H}^0(X;B^{q+1}) \to \check{H}^1(X;Z^q)$$

Because $B^{q+1} \subset \Gamma^{q+1}$, it follows from a similar exactness property that $\check{H}^0(X;B^{q+1}) \subset \check{H}^0(X;\Gamma^{q+1})$. Combining these, we see that

$$\begin{aligned} \check{H}^0(X;Z^q) &\approx \ker\,[\check{H}^0(X;\Gamma^q) \to \check{H}^0(X;B^{q+1})] \\ &\approx \ker\,[\check{H}^0(X;\Gamma^q) \to \check{H}^0(X;\Gamma^{q+1})] \end{aligned}$$

and also that

$$\operatorname{im}\,[\check{H}^0(X;\Gamma^q) \to \check{H}^0(X;\Gamma^{q+1})] \approx \ker\,[\check{H}^0(X;B^{q+1}) \to \check{H}^1(X;Z^q)]$$

Since

$$H^q(\hat{\Gamma}^*(X)) = \ker\,[\check{H}^0(X;\Gamma^q) \to \check{H}^0(X;\Gamma^{q+1})]/\operatorname{im}\,[\check{H}^0(X;\Gamma^{q-1}) \to \check{H}^0(X;\Gamma^q)]$$

the result follows. ■

6 COROLLARY *Let Γ^* be a cochain complex of presheaves of modules on a paracompact Hausdorff space X. For any q there is a short exact sequence, functorial in Γ^*,*

$$0 \to \operatorname{im}[\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})] \to H^q(\hat{\Gamma}^*(X)) \to \\ \ker[\check{H}^0(X;H^q) \to \check{H}^1(X;B^q)] \to 0$$

If Γ^{q-1} is fine, this becomes

$$0 \to \check{H}^1(X;Z^{q-1}) \to H^q(\hat{\Gamma}^*(X)) \to \ker[\check{H}^0(X;H^q) \to \check{H}^1(X;B^q)] \to 0$$

PROOF From the short exact sequence of presheaves

$$0 \to B^q \to Z^q \to H^q \to 0$$

it follows, by theorem 6.7.15, that there is an isomorphism

$$\check{H}^0(X;Z^q)/\check{H}^0(X;B^q) \approx \ker[\check{H}^0(X;H^q) \to \check{H}^1(X;B^q)]$$

From lemma 5, there is an isomorphism

$$\check{H}^0(X;Z^q)/\ker[\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})] \approx H^q(\hat{\Gamma}^*(X))$$

It follows that $H^q(\hat{\Gamma}^*(X))$ maps epimorphically to $\ker[\check{H}^0(X;H^q) \to \check{H}^1(X;B^q)]$ with kernel isomorphic to

$$\check{H}^0(X;B^q)/\ker[\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})] \approx \operatorname{im}[\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})]$$

This gives the first short exact sequence. For the second, there is a short exact sequence of presheaves

$$0 \to Z^{q-1} \to \Gamma^{q-1} \to B^q \to 0$$

and if Γ^{q-1} is fine, it follows from theorems 6.7.15 and 4 that

$$\operatorname{im}[\check{H}^0(X;B^q) \to \check{H}^1(X;Z^{q-1})] = \check{H}^1(X;Z^{q-1}) \quad \blacksquare$$

7 THEOREM *Let Γ^* be a nonnegative cochain complex of fine presheaves of modules on a paracompact Hausdorff space X. Assume that for some integers $0 \le m < n$, $H^q(\Gamma^*)$ is locally zero for $q < m$ and $m < q < n$. Then there are functorial isomorphisms*

$$\check{H}^{q-m}(X;H^m(\Gamma^*)) \approx H^q(\hat{\Gamma}^*(X)) \qquad q < n$$

and a functorial monomorphism

$$\check{H}^{n-m}(X;H^m(\Gamma^*)) \to H^n(\hat{\Gamma}^*(X))$$

PROOF For each q there is a short exact sequence of presheaves

$$0 \to Z^q \to \Gamma^q \to B^{q+1} \to 0$$

Because Γ^q is fine, it follows from theorems 6.7.15 and 4 that

$$(a) \qquad \check{H}^p(X;B^{q+1}) \approx \check{H}^{p+1}(X;Z^q) \qquad p \ge 1$$

For each q there is also a short exact sequence of presheaves

$$0 \to B^q \to Z^q \to H^q \to 0$$

Because H^q is locally zero for $q < m$ and $m < q < n$, it follows from theorems 6.7.15 and 6.7.16 that

$$(b) \qquad \check{H}^p(X;B^q) \approx \check{H}^p(X;Z^q) \qquad q < m \text{ or } m < q < n, \text{ all } p$$

Since B^0 is the zero presheaf, it follows by induction on q from equations (b) and (a) that for $q < m$

$$(c) \qquad \check{H}^p(X;Z^q) = 0 = \check{H}^p(X;B^{q+1}) \qquad p \geq 1$$

From this and corollary 6, it follows that $H^i(\hat{\Gamma}^*(X)) = 0$ for $i < m$. Hence the theorem holds for $q < m$ (both modules being trivial). For $q = m$ we have [by corollary 6 and equation (c)]

$$H^m(\hat{\Gamma}^*(X)) \approx \check{H}^0(X;H^m)$$

and the theorem holds in this case too.

To obtain the result for $m < q \leq n$, note that, by equation (c), $\check{H}^p(X;B^m) = 0$, if $p \geq 1$. From the short exact sequence of presheaves

$$0 \to B^m \to Z^m \to H^m \to 0$$

it follows that

$$\check{H}^p(X;Z^m) \approx \check{H}^p(X;H^m) \qquad p \geq 1$$

For $m < i < n$ it follows from corollary 6 that

$$\check{H}^1(X;Z^{i-1}) \approx H^i(\hat{\Gamma}^*(X))$$

and for $i = n$ there is a monomorphism

$$\check{H}^1(X;Z^{n-1}) \to H^n(\hat{\Gamma}^*(X))$$

Using equations (b) and (a), we see that for $m < i \leq n$

$$\check{H}^1(X;Z^{i-1}) \approx \check{H}^1(X;B^{i-1}) \approx \check{H}^2(X;Z^{i-2}) \approx \cdots \approx \check{H}^{i-m}(X;Z^m) \approx \check{H}^{i-m}(X;H^m)$$

and this gives the result for $m < q \leq n$. ∎

This last result has as an immediate consequence the following isomorphism between the Čech and Alexander cohomologies with coefficients G.

8 COROLLARY *For any paracompact Hausdorff space and module G there is a functorial isomorphism*

$$\check{H}^*(X;G) \approx \bar{H}^*(X;G)$$

of the Čech and Alexander cohomology modules.

PROOF Let C^* be the Alexander presheaf of X with coefficients G. Since C^q is fine for all q (by example 1), this is a nonnegative cochain complex of fine sheaves. Furthermore, for any nonempty U, by lemma 6.4.1,

$$H^q(C^*(U;G)) \approx \begin{cases} 0 & q \neq 0 \\ G & q = 0 \end{cases}$$

Therefore $H^q(C^*)$ is locally zero for $q > 0$ and $H^0(C^*)$ is isomorphic to the constant presheaf G. The hypotheses of theorem 7 are satisfied with $m = 0$ and any n, and there is a functorial isomorphism

$$\check{H}^q(X;G) \approx H^q(\hat{C}^*)$$

for all q. As pointed out in example 6.7.8, there is a canonical isomorphism $\bar{C}^* \approx \hat{C}^*$, and so $\bar{H}^q(X;G) \approx H^q(\hat{C}^*)$. Combining these isomorphisms yields the result. ■

The last result is also true without the assumption of paracompactness (see exercise 6.D.3). The next result is the main uniqueness theorem of the cohomology of presheaves.

9 THEOREM *Let X be a paracompact Hausdorff space and let $\tau: \Gamma^* \to \Gamma'^*$ be a cochain map between nonnegative cochain complexes of fine presheaves of modules on X. Assume that for some $n \geq 0$, $\tau_*: H^q(\Gamma^*) \to H^q(\Gamma'^*)$ is a local isomorphism for $q < n$ and a local monomorphism for $q = n$. Then the induced map*

$$\hat{\tau}_*: H^q(\hat{\Gamma}^*(X)) \to H^q(\hat{\Gamma}'^*(X))$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

PROOF Let Γ_τ^* be the mapping cone of τ (defined for cochain complexes analogous to the definition in Sec. 4.2 for chain complexes). Then $\Gamma_\tau{}^q = \Gamma^{q+1} \oplus \Gamma'^q$, and for $\gamma \in \Gamma^{q+1}(U)$ and $\gamma' \in \Gamma'^q(U)$, $\delta(\gamma,\gamma') = (-\delta(\gamma), \tau(\gamma) + \delta(\gamma'))$. Γ_τ^* is a nonnegative cochain complex of fine presheaves on X, and for any open $U \subset X$ there is an exact sequence

$$\cdots \to H^q(\Gamma'^*(U)) \to H^q(\Gamma_\tau^*(U)) \to H^{q+1}(\Gamma^*(U)) \xrightarrow{\tau_*} H^{q+1}(\Gamma'^*(U)) \to \cdots$$

Taking the direct limit as U varies over open neighborhoods of $x \in X$, we see that $\tau_*: H^q(\Gamma^*) \to H^q(\Gamma'^*)$ is a local isomorphism for $q < n$ and a local monomorphism for $q = n$ if and only if $H^q(\Gamma_\tau^*)$ is locally zero for $q < n$. By theorem 7, it follows that $H^q(\hat{\Gamma}_\tau^*(X)) = 0$ for $q < n$ (if $n = 0$ this is trivially true, and if $n > 0$ it follows from theorem 7 with $m = 0$).

It is obvious that $\hat{\Gamma}_\tau^*$ is the mapping cone $\Gamma_{\hat{\tau}}^*$ of the induced map $\hat{\tau}: \hat{\Gamma}^* \to \hat{\Gamma}'^*$ between the completions. Therefore

$$\cdots \to H^q(\hat{\Gamma}'^*(X)) \to H^q(\hat{\Gamma}_\tau^*(X)) \to H^{q+1}(\hat{\Gamma}^*(X)) \xrightarrow{\hat{\tau}_*} H^{q+1}(\hat{\Gamma}'^*(X)) \to \cdots$$

Since $H^q(\hat{\Gamma}_\tau^*(X))$ was shown to be zero for $q < n$ in the first paragraph above, the result follows from the exactness of this sequence. ■

For compact spaces there is the following *universal-coefficient formula for Čech cohomology.*

10 THEOREM *Let X be a compact Hausdorff space. On the product category* *of presheaves* Γ *on* X *consisting of torsion free* R *modules and the category of* R *modules* G *there is a functorial short exact sequence*

$$0 \to \check{H}^q(X;\Gamma) \otimes G \to \check{H}^q(X; \Gamma \otimes G) \to \check{H}^{q+1}(X;\Gamma) * G \to 0$$

PROOF Let $\mathcal{U}$ be a finite open covering of X. The cochain map

$$\tau: C^*(\mathcal{U};\Gamma) \otimes G \to C^*(\mathcal{U}; \Gamma \otimes G)$$

defined by $\tau(\psi \otimes g)(U_0, \ldots, U_q) = \psi(U_0, \ldots, U_q) \otimes g$ is an isomorphism (this is a consequence of the finiteness of $\mathcal{U}$ analogous to lemma 5.5.6). From the universal-coefficient formula for cochain complexes (theorem 5.4.1), there is a functorial short exact sequence

$$0 \to H^q(\mathcal{U};\Gamma) \otimes G \to H^q(\mathcal{U}; \Gamma \otimes G) \to H^{q+1}(\mathcal{U};\Gamma) * G \to 0$$

The result follows by taking direct limits over the cofinal family of finite open coverings of X (because the tensor product and the torsion product both commute with direct limits). ■

From corollary 8, this gives a universal-coefficient formula for Alexander cohomology of compact spaces. The following theorem generalizes this result to compact pairs and includes the statement that the short exact sequence in question is split.

11 THEOREM *On the product category of pairs* (X,A), *where* A *is a closed subset of a compact Hausdorff space* X, *and the category of* R *modules* G, *there is a functorial short exact sequence*

$$0 \to \bar{H}^q(X,A; R) \otimes G \to \bar{H}^q(X,A; G) \to \bar{H}^{q+1}(X,A; R) * G \to 0$$

and this sequence is split.

PROOF Let $\tau: C^*(\cdot, \cdot \cap A; R) \otimes G \to C^*(\cdot, \cdot \cap A; G)$ be the homomorphism of presheaves defined as in example 6.7.5 [that is, $\tau(\varphi \otimes g)(x_0, \ldots, x_q) = \varphi(x_0, \ldots, x_q)g$]. Both $C^*(\cdot, \cdot \cap A; R) \otimes G$ and $C^*(\cdot, \cdot \cap A; G)$ are nonnegative cochain complexes of fine presheaves. First we prove that

$$\tau_*: H^*(C^*(\cdot, \cdot \cap A; R) \otimes G) \to H^*(C^*(\cdot, \cdot \cap A; G))$$

is a local isomorphism. If $U \subset X - A$, $C^*(U, U \cap A; R) = C^*(U;R)$, and $C^*(U, U \cap A; G) = C^*(U;G)$, it follows from lemma 6.4.1 and theorem 5.4.1 that

$$\tau_*: H^*(C^*(U, U \cap A; R) \otimes G) \approx H^*(C^*(U, U \cap A; G))$$

Since A is closed in X, for any $x \in X - A$, τ_* is an isomorphism of $\varinjlim \{H^*(C^*(U, U \cap A; R) \otimes G)\}$ onto $\varinjlim \{H^*(C^*(U, U \cap A; G))\}$, both limits as U varies over open neighborhoods of x in X.

For any U intersecting A there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & C^*(U, U \cap A; R) \otimes G & \to & C^*(U;R) \otimes G & \to & \bar{C}^*(U \cap A; R) \otimes G & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & C^*(U, U \cap A; G) & \to & C^*(U;G) & \to & \bar{C}^*(U \cap A; G) & \to 0
\end{array}$$

By lemma 6.4.1, the middle cochain complexes have trivial reduced modules. Therefore there is a commutative square

$$\begin{array}{ccc}
\tilde{H}^q(\bar{C}^*(U \cap A; R) \otimes G) & \xrightarrow{\approx} & H^{q+1}(C^*(U, U \cap A; R) \otimes G) \\
\downarrow & & \downarrow \\
\tilde{H}^q(\bar{C}^*(U \cap A; G)) & \xrightarrow{\approx} & H^{q+1}(C^*(U, U \cap A; G))
\end{array}$$

To complete the proof that τ_* is a local isomorphism, therefore, we need only prove that for $x \in A$

$$\lim_{\to} \{\tilde{H}^q(\bar{C}^*(U \cap A; R) \otimes G)\} \approx \lim_{\to} \{\tilde{H}^q(\bar{C}^*(U \cap A; G))\}$$

as U varies over neighborhoods of x in X. This is equivalent to the condition that

$$\tilde{H}^q(\lim_{\to} \{\bar{C}^*(U \cap A; R)\}) \otimes G \approx \tilde{H}^q(\lim_{\to} \{\bar{C}^*(U \cap A; G)\})$$

where $U \cap A$ varies over neighborhoods of x in A. This is trivially true because both sides are zero for all q (this follows from the tautness property of x in the paracompact space A but can be proved without assuming the paracompactness of A, because any one-point subspace is taut in any space with respect to Alexander cohomology).

We have verified that τ satisfies the hypotheses of theorem 9 for all n. Therefore τ induces an isomorphism

$$\hat{\tau}_*: H^*([\hat{C}^*(\cdot, \cdot \cap A; R) \otimes G](X)) \approx H^*(\hat{C}^*(\cdot, \cdot \cap A; G)(X))$$

By example 6.7.8, the right-hand side is isomorphic to $H^*(\bar{C}^*(X,A; G))$. By example 6.7.13, the left hand side is the qth cohomology module of $\check{H}^0(X;C^*(\cdot, \cdot \cap A; R) \otimes G)$. By theorem 10 and the fineness of $C^*(\cdot, \cdot \cap A; R) \otimes G$, this is isomorphic to

$$\begin{aligned}
\check{H}^0(X;C^*(\cdot, \cdot \cap A; R)) \otimes G &\approx (\hat{C}^*(\cdot, \cdot \cap A; R)(X)) \otimes G \\
&\approx \bar{C}^*(X,A; R) \otimes G
\end{aligned}$$

It follows that the map

$$\bar{\tau}: \bar{C}^*(X,A; R) \otimes G \to \bar{C}^*(X,A; G)$$

induced by τ induces an isomorphism of cohomology. The result now follows from the universal-coefficient formula for cochain complexes (theorem 5.4.1). ∎

This implies the following *universal-coefficient formula for Alexander cohomology with compact supports.*

12 COROLLARY *On the product category of pairs (X,A), where A is a closed subset of a locally compact Hausdorff space X, and the category of R modules G, there is a functorial short exact sequence*

$$0 \to \bar{H}_c^q(X,A;\, R) \otimes G \to \bar{H}_c^q(X,A;\, G) \to \bar{H}_c^{q+1}(X,A;\, R) * G \to 0$$

and this sequence is split.

PROOF Let N be a closed cobounded neighborhood of A in X. There is a commutative square of cochain maps

$$\begin{array}{ccc} \bar{C}^*(X,N;\, R) \otimes G & \to & \bar{C}^*(X,N;\, G) \\ \downarrow & & \downarrow \\ \bar{C}^*(\overline{X - N}, \overline{X - N} \cap N;\, R) \otimes G & \to & \bar{C}^*(\overline{X - N}, \overline{X - N} \cap N;\, G) \end{array}$$

in which, by theorem 6.6.5, each vertical map induces an isomorphism of cohomology. By theorem 11, the bottom horizontal map induces an isomorphism of cohomology. Therefore the top horizontal map also induces an isomorphism of cohomology.

There is also a commutative square (in which the limit is over closed cobounded neighborhoods N of A in X)

$$\begin{array}{ccc} \varinjlim \{\bar{C}^*(X,N;\, R)\} \otimes G & \to & \varinjlim \{\bar{C}^*(X,N;\, G)\} \\ \downarrow & & \downarrow \\ \bar{C}_c^*(X,A;\, R) \otimes G & \to & \bar{C}_c^*(X,A;\, G) \end{array}$$

It follows from the first paragraph above that the top horizontal map induces an isomorphism of cohomology. Since the closed cobounded neighborhoods of A in X are cofinal in the family of all cobounded neighborhoods of A in X, it follows from theorem 6.6.15 that each vertical map induces an isomorphism in cohomology. Therefore the bottom horizontal map induces an isomorphism in cohomology. The result follows from this and theorem 5.4.1. ∎

9 APPLICATIONS OF THE COHOMOLOGY OF PRESHEAVES

This section is devoted to two main applications of the theory developed in the last two sections. One is the study of the relation between Alexander and singular cohomology. We shall prove that in a homologically locally connected space (for example, a manifold) the two are isomorphic. The other application is to a study of the relation between the Alexander cohomology of two spaces connected by a continuous map. We conclude with a proof of the Vietoris-Begle mapping theorem.

Let (X,A) be a pair and let G be an R module. Recall the homomorphism

$$\tau\colon C^*(\cdot\,,\, \cdot \cap A;\, G) \to \Delta^*(\cdot\,,\, \cdot \cap A;\, G)$$

defined in example 6.7.4. This induces a homomorphism

$$\hat{\tau}\colon \hat{C}^*(\cdot, \cdot \cap A; G) \to \hat{\Delta}^*(\cdot, \cdot \cap A; G)$$

such that the following square is commutative

$$\begin{array}{ccc} C^*(\cdot, \cdot \cap A; G) & \xrightarrow{\tau} & \Delta^*(\cdot, \cdot \cap A; G) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \hat{C}^*(\cdot, \cdot \cap A; G) & \xrightarrow{\hat{\tau}} & \hat{\Delta}^*(\cdot, \cdot \cap A; G) \end{array}$$

By examples 6.7.8 and 6.7.9, there are isomorphisms

$$\bar{C}^*(\cdot, \cdot \cap A; G) \approx \hat{C}^*(\cdot, \cdot \cap A; G)$$
$$\alpha_*\colon H^*(\Delta^*(\cdot, \cdot \cap A; G)) \approx H^*(\hat{\Delta}^*(\cdot, \cdot \cap A; G))$$

In Sec. 6.5 a natural homomorphism

$$\mu\colon \bar{H}^*(X,A; G) \to H^*(X,A; G)$$

was defined, and it is a simple matter to check that commutativity holds in the diagram

$$\begin{array}{ccc} H^*(\bar{C}^*(X,A; G)) & \xrightarrow{\mu} & H^*(\Delta^*(X,A; G)) \\ \approx\downarrow & & \approx\downarrow{\scriptstyle\alpha_*} \\ H^*(\hat{C}^*(\cdot, \cdot \cap A; G)(X)) & \xrightarrow{\hat{\tau}_*} & H^*(\hat{\Delta}^*(\cdot, \cdot \cap A; G)(X)) \end{array}$$

Therefore μ is an isomorphism if and only if $\hat{\tau}_*$ is.

1 THEOREM *Let X be a paracompact Hausdorff space and suppose there is $n \geq 0$ such that each $x \in X$ is taut with respect to singular cohomology with coefficients G in degrees $< n$. Then*

$$\mu\colon \bar{H}^q(X;G) \to H^q(X;G)$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

PROOF Both $C^*(\cdot\,;G)$ and $\Delta^*(\cdot\,;G)$ are nonnegative cochain complexes of fine presheaves. The tautness assumption of the points of X with respect to singular cohomology implies that $\tau_*\colon H^q(C^*(\cdot\,;G)) \to H^q(\Delta^*(\cdot\,;G))$ is a local isomorphism for $q < n$ and a local monomorphism for $q = n$ (in fact, it is always a local monomorphism for all q). By theorem 6.8.9,

$$\hat{\tau}_*\colon H^q(\hat{C}^*(X;G)) \to H^q(\hat{\Delta}^*(X;G))$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$. ∎

There is a partial converse of theorem 1 which asserts that if $\mu\colon \bar{H}^q(U;G) \to H^q(U;G)$ is an isomorphism for $q < n$ and every open $U \subset X$, then each point $x \in X$ is taut with respect to singular cohomology in degrees $< n$. This follows from commutativity of the following diagram (where U varies over open neighborhoods of $x \in X$):

$$\begin{array}{ccc}
\lim_{\to} \{\bar{H}^q(U;G)\} & \xrightarrow{\approx} & \bar{H}^q(x;G) \\
{\scriptstyle\mu}\downarrow & & \approx\downarrow{\scriptstyle\mu} \\
\lim_{\to} \{H^q(U;G)\} & \to & H^q(x;G)
\end{array}$$

In case X is a Hausdorff space in which every open subset is paracompact (for example, X is metrizable), we see that each point $x \in X$ is taut with respect to singular cohomology in degrees $< n$ if and only if μ: $\bar{H}^q(U;G) \to H^q(U;G)$ is an isomorphism for all $q < n$ and all open $U \subset X$.

A space X is said to be *homologically locally connected in dimension n* if for every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that $\tilde{H}_q(V) \to \tilde{H}_q(U)$ is trivial for $q \le n$. It is said to be *homologically locally connected* if it is homologically locally connected in dimension n for all n.

2 EXAMPLE Any locally contractible space, in particular any polyhedron or any manifold, is homologically locally connected in dimension n for all n.

3 EXAMPLE Let $X_q = S^q$ for $q \ge 1$ and let x_q be a base point of X_q. The subspace of $\times X_q$ consisting of all points having at most one coordinate different from the corresponding base point is homologically locally connected in dimension n for all n but is not locally contractible.

4 LEMMA *If X is homologically locally connected in dimension n, then $\tilde{H}^q(\Delta^*(\cdot\,;G))$ is locally zero for $q \le n$ and all G.*

PROOF Let $c^* \in \operatorname{Hom}(\tilde{\Delta}_q(U),G)$ be a cocycle $(0 \le q \le n)$ and let $x \in U$. If $q = 0$, let V be a neighborhood of x in U such that $\tilde{H}_0(V) \to \tilde{H}_0(U)$ is trivial. If $c \in \tilde{\Delta}_0(V)$, there is $c' \in \Delta_1(U)$ such that $c = \partial c'$. Then $c^*(c) = c^*(\partial c') = (\delta c^*)(c') = 0$. Therefore $c^* \mid \tilde{\Delta}_0(V) = 0$, proving that $\tilde{H}^0(\Delta^*(\cdot\,;G))$ is locally trivial.

If $0 < q$, let V and V' be neighborhoods of x in U, with $V \subset V'$ and such that $\tilde{H}_{q-1}(V) \to \tilde{H}_{q-1}(V')$ and $H_q(V') \to H_q(U)$ are both trivial. If c is a reduced singular $(q-1)$-cycle of V, let c' be a q-chain of V' such that $\partial c' = c$. Then $c^*(c') \in G$ is independent of the choice of c'; if c'' is another q-chain in V' such that $\partial c'' = c$, then $c' - c'' = \partial d$ for some $(q+1)$-chain d in U and

$$c^*(c' - c'') = c^*(\partial d) = (\delta c^*)(d) = 0$$

Hence there is a homomorphism $\bar{c}^*: \tilde{Z}_{q-1}(V) \to G$ such that $\bar{c}^*(c) = c^*(c')$ if $\partial c' = c$. Because $\Delta_{q-1}(V)/\tilde{Z}_{q-1}(V)$ is free (since it is isomorphic to a subgroup of $\Delta_{q-2}(V)$ if $q > 1$ or to $\mathbf{Z}$ if $q = 1$), there is a homomorphism $d^*: \Delta_{q-1}(V) \to G$ which is an extension of $\bar{c}^*$. Then $c^* \mid \Delta_q(V) = \delta d^*$, proving that $H^q(\Delta^*(\cdot\,;G))$ is locally trivial. ∎

5 COROLLARY *If X is a paracompact Hausdorff space homologically locally connected in dimension n, then μ: $\bar{H}^q(X;G) \to H^q(X;G)$ is an isomorphism for $q \le n$ and a monomorphism for $q = n + 1$.* ∎

6 COROLLARY *Let A be a closed subset, homologically locally connected in dimension n, of a Hausdorff space X, homologically locally connected in dimension n. If X has the property that every open subset is paracompact,* $\mu\colon \bar{H}_c^q(X,A;\, G) \to H_c^q(X,A;\, G)$ *is an isomorphism for* $q \le n$ *and a monomorphism for* $q = n + 1$.

PROOF From the definitions, there is a commutative square (where U varies over open cobounded subsets of X)

$$\begin{array}{ccc} \lim_{\to} \{\bar{H}^q(X,U;\, G)\} & \xrightarrow{\approx} & \bar{H}_c^q(X;G) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \lim_{\to} \{H^q(X,U;\, G)\} & \xrightarrow{\approx} & H_c^q(X;G) \end{array}$$

Since an open subset of a space homologically locally connected in dimension n is again a space homologically locally connected in dimension n corollary 5 applies to X and to every open $U \subset X$. By the five lemma,

$$\mu\colon \bar{H}^q(X,U;\, G) \to H^q(X,U;\, G)$$

is an isomorphism for $q \le n$ and a monomorphism for $q = n + 1$. Passing to the limit, $\mu\colon \bar{H}_c^q(X;G) \to H_c^q(X;G)$ is an isomorphism for $q \le n$ and a monomorphism for $q = n + 1$. Since A has the same properties as X,

$$\mu\colon \bar{H}_c^q(A;G) \to H_c^q(A;G)$$

is an isomorphism for $q \le n$ and a monomorphism for $q = n + 1$. The result now follows from the five lemma. ∎

Since a manifold is homologically locally connected in dimension n for all n, and every open subset is paracompact, this implies the next result.

7 COROLLARY *If X is a manifold,* $\mu\colon \bar{H}^*(X;G) \approx H^*(X;G)$. *If A is a closed homologically locally connected subset of X,*

$$\mu\colon \bar{H}_c^*(X,A;\, G) \approx H_c^*(X,A;\, G). \quad ∎$$

8 COROLLARY *If X is a homologically locally connected space imbedded as a closed subset of a manifold Y, then X is taut in Y with respect to singular cohomology.*

PROOF By corollary 5, $\bar{H}^*(X;G) \approx H^*(X;G)$, and for an open set U in Y, by corollary 7, $\bar{H}^*(U;G) \approx H^*(U;G)$. Since X is taut in Y with respect to Alexander cohomology, these isomorphisms imply that it is also taut with respect to singular cohomology. ∎

9 COROLLARY *If A is any closed subset of a manifold X, then as U varies over neighborhoods of A in X,*

$$\lim_{\to} \{H^*(U;G)\} \approx \bar{H}^*(A;G)$$

where the right-hand side is Alexander cohomology.

PROOF By corollary 7, $\lim_{\to} \{\bar{H}^*(U;G)\} \approx \lim_{\to} \{H^*(U;G)\}$, so the result follows from the tautness of A with respect to the Alexander cohomology theory. ∎

This shows that the modules $\bar{H}^*(A;G)$ and $\bar{H}^*(A,B;\ G)$ introduced in Sec. 6.1 are the Alexander cohomology modules if A [or (A,B)] is a closed subset [or pair] of a manifold. The next result generalizes the duality theorem 6.2.17 to arbitrary closed pairs.

10 THEOREM *Let X be an n-manifold orientable over R. For any closed pair (A,B) in X and any R module G there is an isomorphism*

$$H_q(X - B,\ X - A;\ G) \approx \bar{H}_c^{n-q}(A,B;\ G)$$

PROOF Let N be a closed cobounded neighborhood of B in A. By theorem 6.6.5, there is an isomorphism

$$\bar{H}^{n-q}(A,N;\ G) \approx \bar{H}^{n-q}(\overline{A - N},\ \overline{A - N} \cap N;\ G)$$

Since $(\overline{A - N},\ \overline{A - N} \cap N)$ is a compact pair in X, by theorem 6.2.17,

$$H_q(X - (\overline{A - N} \cap N),\ X - (\overline{A - N});\ G) \approx \bar{H}^{n-q}(\overline{A - N},\ \overline{A - N} \cap N;\ G)$$

Since $X - (\overline{A - N})$ and $X - N$ are open, there is an excision isomorphism

$$H_q(X - N,\ X - A;\ G) \approx H_q(X - (\overline{A - N} \cap N),\ X - (\overline{A - N});\ G)$$

Combining these gives an isomorphism

$$H_q(X - N,\ X - A;\ G) \approx \bar{H}^{n-q}(A,N;\ G)$$

As N varies over closed cobounded neighborhoods of B in A, the limit of the modules on the left is $H_q(X - B,\ X - A;\ G)$ and the limit of the modules on the right is $\bar{H}_c^{n-q}(A,B;\ G)$, whence the result. ∎

11 THEOREM *If X is a compact Hausdorff space which is homologically locally connected in dimension n, then $H_q(X)$ is finitely generated for $q \leq n$.*

PROOF This follows from corollary 5, theorem 6.8.11, and theorem 5.5.13. ∎

The last result gives a generalization of corollary 6.2.21 to arbitrary compact manifolds (orientable or not). We now work toward a proof of the Vietoris-Begle mapping theorem.

12 LEMMA *Let (X,A) be a pair and let Γ be the presheaf on X defined by $\Gamma(V) = \bar{C}^q(V,\ V \cap A;\ G)$ for open $V \subset X$ (q and G being fixed).*

(*a*) *For any open covering $\mathcal{U}$ of X the map $\Gamma(X) \to \Gamma(\mathcal{U})$ sending $\gamma \in \Gamma(X)$ to the compatible $\mathcal{U}$ family $\{\gamma \mid U\}_{U \in \mathcal{U}}$ is a monomorphism.*
(*b*) *If $\mathcal{U}$ is a locally finite open covering of X and $\mathcal{V}$ is a shrinking of $\mathcal{U}$, the image of $\Gamma(\mathcal{U}) \to \Gamma(\mathcal{V})$ equals the image of the composite*

$$\Gamma(X) \to \Gamma(\mathcal{U}) \to \Gamma(\mathcal{V})$$

PROOF For (a), assume that $\gamma \in \bar{C}^q(X,A;\ G)$ is in the kernel of $\Gamma(X) \to \Gamma(\mathcal{U})$

(that is, $\gamma \mid U = 0$ for all $U \in \mathfrak{U}$). Let $\varphi \in C^q(X,A;\ G)$ be a representative of γ. Then $\gamma \mid U = 0$ implies that $\varphi \mid U$ is locally zero on U. Since this is so for all $U \in \mathfrak{U}$, φ is locally zero on X and $\gamma = 0$, proving (a).

To prove (b), let $\{\gamma_U\}_{U \in \mathfrak{U}}$ be a compatible $\mathfrak{U}$ family and suppose that $\varphi_U \in C^q(U, U \cap A;\ G)$ is a representative of γ_U for $U \in \mathfrak{U}$. Then, for $U, U' \in \mathfrak{U}$, $\varphi_U \mid U \cap U' - \varphi_{U'} \mid U \cap U'$ is locally zero on $U \cap U'$. If $x \in X$, some neighborhood of x meets only finitely many elements of $\mathfrak{U}$, and there is a smaller neighborhood W_x of x such that

$$\begin{array}{ll} \text{(i)} & W_x \text{ intersects } \bar{V}_U \Leftrightarrow x \in \bar{V}_U \\ \text{(ii)} & x \in U \Rightarrow W_x \subset U \\ \text{(iii)} & x \in V_U \Rightarrow W_x \subset V_U \\ \text{(iv)} & x \in \bar{V}_U \cap \bar{V}_{U'} \Rightarrow \varphi_U \mid W_x = \varphi_{U'} \mid W_x \end{array}$$

The first three conditions are clearly satisfied by taking W_x small enough (because there are only a finite number of conditions to be satisfied) and (iv) can also be satisfied, because for $x \in \bar{V}_U \cap \bar{V}_{U'}$, $\varphi_U \mid U \cap U' - \varphi_{U'} \mid U \cap U'$ is locally zero.

For $x \in X$ choose U so that $x \in \bar{V}_U$ and set $\varphi_x = \varphi_U \mid W_x \in C^q(W_x, W_x \cap A;\ G)$. By (iv), this is independent of the choice of U. If $x'' \in W_x \cap W_{x'}$, then $x'' \in \bar{V}_U$ for some $U \in \mathfrak{U}$. Then W_x and $W_{x'}$ meet $\bar{V}_U$, and by (i), x, $x' \in \bar{V}_U$. Therefore $\varphi_x = \varphi_U \mid W_x$ and $\varphi_{x'} = \varphi_U \mid W_{x'}$, whence $\varphi_x \mid W_x \cap W_{x'} = \varphi_{x'} \mid W_x \cap W_{x'}$. Hence the collection $\{\varphi_x \in C^q(W_x, W_x \cap A;\ G)\}$ is a compatible $\{W_x\}$ family [of $C^q(\cdot, \cdot \cap A;\ G)$]. By example 6.7.8, there is an element $\varphi \in C^q(X,A;\ G)$ such that $\varphi \mid W_x = \varphi_x$ for all $x \in X$. To complete the proof of (b) it suffices to prove that for each $U \in \mathfrak{U}$, $\varphi \mid V_U - \varphi_U \mid V_U$ is locally zero on V_U. However, if $x \in V_U$, then, by (iii), $W_x \subset V_U$ and $\varphi \mid W_x = \varphi_x = \varphi_U \mid W_x$. Hence $\{W_x\}_{x \in V_U}$ is an open covering of V_U on which $\varphi \mid V_U$ and $\varphi_U \mid V_U$ agree. ∎

13 THEOREM *Let $f: X' \to X$ be a closed continuous map between paracompact Hausdorff spaces. Let A' be a closed subset of X' and suppose there are integers $0 \le m < n$ such that $\bar{H}^q(f^{-1}x, f^{-1}x \cap A';\ G) = 0$ for all $x \in X$ and for $q < m$ or $m < q < n$. Let Γ be the presheaf on X defined by $\Gamma(U) = \bar{H}^m(f^{-1}(U), f^{-1}(U) \cap A';\ G)$. Then there are isomorphisms*

$$\check{H}^{q-m}(X;\Gamma) \approx \bar{H}^q(X',A';\ G) \qquad q < n$$

and a monomorphism

$$\check{H}^{n-m}(X;\Gamma) \to \bar{H}^n(X',A';\ G)$$

PROOF Let Γ^* be the nonnegative cochain complex of presheaves on X defined by $\Gamma^*(U) = \bar{C}^*(f^{-1}(U), f^{-1}(U) \cap A';\ G)$. Thus Γ^q is the image under f_* of the fine presheaf on X' which assigns $\bar{C}^q(U', U' \cap A';\ G)$ to $U' \subset X'$. By theorem 6.8.3c, the latter is a fine presheaf on X' [being the completion of the fine presheaf $C^q(\cdot, \cdot \cap A';\ G)$; see example 6.8.1], and by theorem 6.8.3b, Γ^q is fine on X. As U varies over neighborhoods of x in X, $(f^{-1}(U), f^{-1}(U) \cap A')$

varies over a cofinal family of neighborhoods of $(f^{-1}x, f^{-1}x \cap A')$ in (X',A') (because f is closed and continuous). From the standard tautness properties and the hypothesis about $\bar{H}^*(f^{-1}x, f^{-1}x \cap A'; G)$, it follows that $H^q(\Gamma^*)$ is locally zero for $q < m$ and $m < q < n$. By theorem 6.8.7, there are functorial isomorphisms

$$\check{H}^{q-m}(X;H^m(\Gamma^*)) \approx H^q(\hat{\Gamma}^*(X)) \qquad q < n$$

and a monomorphism

$$\check{H}^{n-m}(X;H^m(\Gamma^*)) \to H^n(\hat{\Gamma}^*(X))$$

Since $\Gamma = H^m(\Gamma^*)$, it merely remains to verify that

$$H^p(\hat{\Gamma}^*(X)) \approx \bar{H}^p(X',A'; G) \qquad \text{all } p$$

As $\mathfrak{U}$ varies over the cofinal family of locally finite open coverings of X it follows from lemma 12 that

$$\hat{\Gamma}^*(X) = \lim_{\to} \{\Gamma^*(\mathfrak{U})\} = \lim_{\to} \{\bar{C}^*(\cdot, \cdot \cap A'; G)(f^{-1}\mathfrak{U})\} \approx \bar{C}^*(X',A'; G)$$

and this yields the result. ∎

If ξ is an m-sphere bundle over a paracompact Hausdorff base space B, then $\bar{H}^q(p_\xi^{-1}(x), p_\xi^{-1}(x) \cap \dot{E}) = 0$ if $q \neq m + 1$. Therefore the hypotheses of theorem 13 are satisfied for all n. Since the presheaf Γ that occurs in theorem 13 is the tensor product of the orientation presheaf of ξ and G, we obtain the following generalization of the Thom isomorphism theorem to nonorientable sphere bundles.

14 THEOREM *Let ξ be an m-sphere bundle over a paracompact Hausdorff base space B and let Γ be the orientation presheaf of ξ over R. For all R modules G and all q there is an isomorphism*

$$\check{H}^q(B; \Gamma \otimes G) \approx \bar{H}^{q+m+1}(E_\xi,\dot{E}_\xi; G) \quad ∎$$

Another interesting consequence of theorem 13 is the following *Vietoris-Begle mapping theorem.*

15 THEOREM *Let $f: X' \to X$ be a closed continuous surjective map between paracompact Hausdorff spaces. Assume that there is $n \geq 0$ such that $\tilde{\bar{H}}^q(f^{-1}x;G) = 0$ for all $x \in X$ and for $q < n$. Then*

$$f^*: \bar{H}^q(X;G) \to \bar{H}^q(X';G)$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

PROOF Let Z be the mapping cylinder of f and regard X' as imbedded in Z. Then Z is a paracompact Hausdorff space, X' is closed in Z, and the retraction $r: Z \to X$ is a closed continuous map. For $x \in X$, $r^{-1}(x)$ is contractible [since it is homeomorphic to the join of x with $f^{-1}(x)$], and so $\tilde{\bar{H}}^*(r^{-1}(x)) = 0$. Because $r^{-1}(x) \cap X' = f^{-1}(x)$ is nonempty, we have

$$\bar{H}^{q+1}(r^{-1}(x), r^{-1}(x) \cap X'; G) \approx \tilde{\bar{H}}^q(f^{-1}(x); G) = 0 \qquad q < n$$

It follows from theorem 13 that $\bar{H}^q(Z,X'; G) = 0$ for $q \leq n$. Since there is a commutative diagram with an exact row

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{H}^q(Z,X') \arrow[r] & \bar{H}^q(Z) \arrow[r] & \bar{H}^q(X') \arrow[r] & \bar{H}^{q+1}(Z,X') \arrow[r] & \cdots \\
& & \bar{H}^q(X) \arrow[u, "r^*"', "\approx"] \arrow[ur, "f^*"'] & & &
\end{tikzcd}$$

the result follows. ■

There is a partial converse of theorem 15 asserting that if $f\colon X' \to X$ is a closed continuous surjective map between paracompact Hausdorff spaces and there is $n \geq 0$ such that for every open $U \subset X$, $f^*\colon \bar{H}^q(U;G) \to \bar{H}^q(f^{-1}(U);G)$ is an isomorphism for $q < n$, then $\tilde{\bar{H}}^q(f^{-1}(x);G) = 0$ for all $x \in X$ and for $q < n$. This follows from commutativity of the following diagram (where U varies over open neighborhoods of $x \in X$):

$$\begin{tikzcd}
\lim_{\to} \{\tilde{\bar{H}}^q(U;G)\} \arrow[r, "\approx"'] \arrow[d, "f^*"'] & \tilde{\bar{H}}^q(x;G) \arrow[d, "f^*"] \\
\lim_{\to} \{\tilde{\bar{H}}^q(f^{-1}(U);G)\} \arrow[r, "\approx"'] & \tilde{\bar{H}}^q(f^{-1}(x);G)
\end{tikzcd}$$

In particular, if X and X' are metrizable (or have the property that every open subset is paracompact), then for $n \geq 0$, $f^*\colon \bar{H}^q(U;G) \to \bar{H}^q(f^{-1}(U);G)$ is an isomorphism for all open $U \subset X$ and all $q < n$ if and only if $\tilde{\bar{H}}^q(f^{-1}(x);G) = 0$ for all $x \in X$ and all $q < n$.

We present an example to show that the condition that f be a closed map is necessary in theorem 15.

16 EXAMPLE Let $X' = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \text{ or } x^2 + y^2 < 1, x > 0\}$ and let $X = [0,1]$. Define $f\colon X' \to X$ by

$$f(x,y) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$$

Then f is a continuous surjective map but not a closed map. Furthermore,

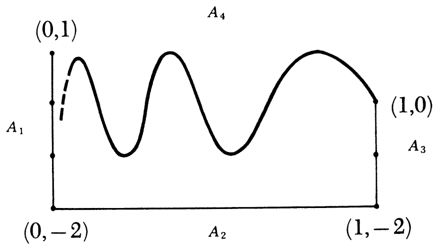
$$f^{-1}(t) = \begin{cases} \text{closed semicircle} & t = 0 \\ \text{closed interval} & 0 < t < 1 \\ \text{single point} & t = 1 \end{cases}$$

Because the unit circle S^1 is a strong deformation retract of X',

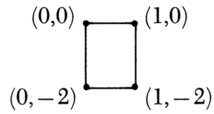
$$\bar{H}^1(X';G) \approx \bar{H}^1(S^1;G) \approx G.$$

Since $\bar{H}^1(X;G) = 0$, the homomorphism $f^*\colon \bar{H}^1(X;G) \to \bar{H}^1(X';G)$ is not an isomorphism.

17 EXAMPLE Let $X \subset \mathbf{R}^2$ be the space of example 2.4.8, illustrated below:



There is a closed continuous surjective map f of X onto the space Y consisting of the four sides of the rectangle



such that

$$f^{-1}(y) = \begin{cases} \text{single point} & y \neq (0,0) \\ \text{closed interval} & y = (0,0) \end{cases}$$

It follows from theorem 15 that $f^*\colon \bar{H}^*(Y;G) \approx \bar{H}^*(X;G)$ for any G, and therefore the map f is not null homotopic.

18 THEOREM *Let $f\colon X' \to X$ be a proper surjective map between locally compact Hausdorff spaces and assume that for some $n > 0$, $\tilde{\bar{H}}^q(f^{-1}(x);G) = 0$ for all $x \in X$ and all $q < n$. Then*

$$f^*\colon \bar{H}_c{}^q(X;G) \to \bar{H}_c{}^q(X';G)$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

PROOF If either X or X' is compact, the other one is also compact, and the result follows from lemma 6.6.9 and theorem 15. If neither X nor X' is compact, let X^+ and X'^+ be their one-point compactifications and extend f to a map $f^+\colon X'^+ \to X^+$ mapping the point at infinity of X'^+ to the point at infinity of X^+. Then f^+ satisfies the hypotheses of theorem 15, and the result follows from corollary 6.6.12 and theorem 15. ∎

10 CHARACTERISTIC CLASSES

This section is a culmination of our general work on homology theory. We use the cup product and Steenrod squares to define characteristic classes of a manifold and of one manifold imbedded in another. These characteristic classes are important invariants of the manifold and have interesting applications to nonimbedding problems.

Let X be an n-manifold with boundary $\dot{X}$ and $U \in H^n(X \times X, X \times X - \delta(X))$

be an orientation (over R) of X. Let $j\colon X - \dot{X} \subset X$ be the inclusion map. Then the maps

$$j \times 1\colon (X - \dot{X}) \times (X,\dot{X}) \subset X \times (X,\dot{X})$$
$$1 \times j\colon (X,\dot{X}) \times (X - \dot{X}) \subset (X,\dot{X}) \times X$$

are both homotopy equivalences. Therefore there are elements

$$U_1 \in H^n(X \times (X,\dot{X})) \qquad U_2 \in H^n((X,\dot{X}) \times X)$$

such that

$$(j \times 1)^* U_1 = U \mid (X - \dot{X}) \times (X,\dot{X}) \qquad (1 \times j)^* U_2 = U \mid (X,\dot{X}) \times (X - \dot{X})$$

If X is compact, let $z \in H_n(X,\dot{X})$ be the fundamental class of X corresponding to U, as in theorem 6.3.9. The *Euler class* of a compact oriented manifold X, denoted by $\chi \in H^n(X,\dot{X})$, is defined by

$$\chi = (U_1 \smile U_2)/z$$

The reason for the name is furnished by theorem 2 below.

Assume that R is a field and that X is a compact n-manifold with boundary $\dot{X}$. By theorem 6.9.11, $H_*(X)$ and $H_*(X,\dot{X})$ are finitely generated. If $\{u_i\}$ is a basis of $H^*(X)$ and $\{v_j\}$ is a basis of $H^*(X,\dot{X})$, then by the Künneth formula for cohomology, $\{u_i \times v_j\}$ is a basis of $H^*(X \times (X,\dot{X}))$. Hence

$$U_1 = \sum_{i,j} a_{ij} u_i \times v_j$$

for some scalars a_{ij}. Let $b_{jk} = \langle v_j \smile u_k, z\rangle$, where z is the fundamental class corresponding to U. Then we have matrices $A = (a_{ij})$ and $B = (b_{jk})$, and the following expresses their relation to each other.

1 LEMMA *With the above notation,*

$$(AB)_{ik} = (-1)^{n \deg u_k} \delta_{ik}$$

PROOF The proof is essentially the same as that for theorem 6.3.12. Because z is the fundamental class corresponding to U, it follows that

$$U_1/z = 1 \in H^0(X)$$

By property 6.1.4, for any k

$$u_k = u_k \smile 1 = u_k \smile U_1/z = [(u_k \times 1) \smile U_1]/z$$

From lemma 6.3.11 it follows readily that

$$\begin{aligned}(u_k \times 1) \smile U_1 &= (1 \times u_k) \smile U_1 = (-1)^{n \deg u_k} U_1 \smile (1 \times u_k)\\ &= \sum_{i,j} (-1)^{n \deg u_k} a_{ij} u_i \times (v_j \smile u_k)\end{aligned}$$

Hence by property 6.1.2

$$u_k = \sum_{i,j} (-1)^{n \deg u_k} a_{ij} b_{jk} u_i$$

Since $\{u_i\}$ is a basis, this implies the result. ∎

2 THEOREM *If χ is the Euler class of a compact n-manifold X oriented over a field, then $\langle \chi, z\rangle$ is the Euler characteristic of X.*

PROOF We first compute U_2. Let $T: X \times X \to X \times X$ be the map interchanging the factors. There is a commutative diagram, with all vertical maps induced by maps defined by T and all horizontal maps induced by inclusions,

$$\begin{array}{ccccc} H^n(X \times X, X \times X - \delta(X)) & \to & H^n((X - \dot X) \times (X,\dot X)) & \xleftarrow[\approx]{(j \times 1)^*} & H^n(X \times (X,\dot X)) \\ {\scriptstyle T_1^*}\downarrow & & {\scriptstyle T_2^*}\downarrow & & \downarrow{\scriptstyle T_3^*} \\ H^n(X \times X, X \times X - \delta(X)) & \to & H^n((X,\dot X) \times (X - \dot X)) & \xleftarrow[\approx]{(1 \times j)^*} & H^n((X,\dot X) \times X) \end{array}$$

In the proof of lemma 6.3.11 it was shown that $T_1^*\ U = (-1)^n U$. Therefore $T_3^*\ U_l = (-1)^n U_2$, and so

$$\begin{aligned} U_2 &= (-1)^n T_3^* \Big(\sum_{k,l} a_{kl} u_k \times v_l \Big) \\ &= (-1)^n \sum_{k,l} (-1)^{\deg u_k \deg v_l} a_{kl} v_l \times u_k \end{aligned}$$

Therefore

$$\begin{aligned} U_1 \smile U_2 &= (-1)^n \sum (-1)^{\deg v_l \deg v_j + \deg u_k \deg v_l} a_{ij} a_{kl} (u_i \smile v_l) \times (v_j \smile u_k) \\ &= (-1)^n \sum (-1)^{\deg v_l \deg v_j + \deg u_k \deg v_l + \deg u_i \deg v_l} a_{ij} a_{kl} (v_l \smile u_i) \\ &\qquad \times (v_j \smile u_k) \end{aligned}$$

where the summation is over all i, j, k, and l such that

$$\deg u_i + \deg v_j = n = \deg u_k + \deg v_l$$

It follows that

$$U_1 \smile U_2 = \sum (-1)^{\deg u_k} a_{ij} a_{kl} (v_l \smile u_i) \times (v_j \smile u_k)$$

Using lemma 1,

$$\begin{aligned} \langle \chi, z\rangle &= \langle U_1 \smile U_2, z \times z\rangle \\ &= \sum_{i,j,k,l} (-1)^{\deg u_k} a_{ij} b_{jk} a_{kl} b_{li} \\ &= \sum_{i,k} (-1)^{\deg u_k} (AB)_{ik}\ (AB)_{ki} \\ &= \sum_k (-1)^{\deg u_k} \end{aligned}$$

and the last sum is the Euler characteristic of X. ■

Classically, the Euler class is usually taken to be the Euler class (in our sense) over $\mathbf{Z}$. For any pair (Y,B) whose homology is of finite type, it follows from the universal-coefficient formula for cohomology (theorem 5.5.10) that

$$H^q(Y,B;\ \mathbf{R}) \approx H^q(Y,B;\ \mathbf{Z}) \otimes \mathbf{R}$$

Therefore the monomorphism $\mathbf{Z} \to \mathbf{R}$ induces a monomorphism

$$H^q(Y,B;\ \mathbf{Z}) \to H^q(Y,B;\ \mathbf{R})$$

In particular, the monomorphism $H^n(X,\dot{X};\, \mathbf{Z}) \to H^n(X,\dot{X};\, \mathbf{R})$ maps Euler class to Euler class, and therefore theorem 2 remains valid for the integral Euler class of X.

We now specialize to the case where the coefficient field is $\mathbf{Z}_2$, in which case U, hence also U_1, and (if X is compact) z, are all unique. There is the Thom isomorphism

$$\Phi^*: H^q(X - \dot{X}) \approx H^{q+n}((X - \dot{X}) \times (X - \dot{X}), (X - \dot{X}) \times (X - \dot{X}) - \delta(X - \dot{X}))$$

defined by $\Phi^*(v) = (v \times 1) \smile U'$, where

$$U' = U \,|\, ((X - \dot{X}) \times (X - \dot{X}), (X - \dot{X}) \times (X - \dot{X}) - \delta(X - \dot{X}))$$

Φ^* can be extended to

$$\Phi^*: H^q(X) \to H^{q+n}(X \times X, X \times X - \delta(X))$$

by $\Phi^*(v) = (v \times 1) \smile U$. There is a commutative diagram whose vertical maps are isomorphisms

$$\begin{array}{ccc} H^q(X) & \xrightarrow{\Phi^*} & H^{q+n}(X \times X, X \times X - \delta(X)) \\ \approx\downarrow & & \approx\downarrow \\ H^q(X - \dot{X}) & \xrightarrow{\Phi^*} & H^{q+n}((X - \dot{X}) \times (X - \dot{X}), (X - \dot{X}) \times (X - \dot{X}) - \delta(X - \dot{X})) \end{array}$$

from which it follows that Φ^* is also an isomorphism on $H^q(X)$. For $i \geq 0$ the *ith Stiefel-Whitney class of* X, $w_i \in H^i(X;\mathbf{Z}_2)$, is defined by the formula

$$\Phi^*(w_i) = Sq^i U$$

[that is, $Sq^i U = (w_i \times 1) \smile U$]. Following are some examples.

3 By condition (*a*) on page 271, $w_0 = 1$.

4 By condition (*b*) on page 271, if X is a compact n-manifold without boundary, w_n is the Euler class of X over $\mathbf{Z}_2$.

5 By condition (*c*) on page 271, $w_i = 0$ for $i > \dim X$.

6 A manifold X is orientable over $\mathbf{Z}$ if and only if $w_1 = 0$ (see exercise 5.H.3*d*).

If X is compact and $z \in H_n(X,\dot{X})$ is the fundamental class of X over $\mathbf{Z}_2$, then, by property 6.1.4,

$$w_i = [(w_i \times 1) \smile U_1]/z = Sq^i U_1/z$$

where $U_1 \in H^n(X \times (X,\dot{X}))$ corresponds to U. We use this to determine the Stiefel-Whitney classes of a compact X in terms of cohomology operations in X. For $i \geq 0$ the homomorphism $Sq^i\colon H^{n-i}(X,\dot{X}) \to H^n(X,\dot{X})$ has a transpose homomorphism $\overline{Sq^i}\colon H_n(X,\dot{X}) \to H_{n-i}(X,\dot{X})$ such that

$$\langle Sq^i u, z\rangle = \langle u, \overline{Sq^i} z\rangle \qquad u \in H^{n-i}(X,\dot{X})$$

where z is the fundamental class of X. By the isomorphism of theorem 6.3.12,

$$\kappa_z\colon H^i(X) \approx H_{n-i}(X,\dot{X})$$

and there is a unique $V_i \in H^i(X)$ such that $\kappa_z(V_i) = \overline{Sq}^i(z)$. Then for $u \in H^{n-i}(X,\dot{X}; \mathbf{Z}_2)$

$$\begin{aligned}\langle Sq^i u,z\rangle &= \langle u,\overline{Sq}^i z\rangle = \langle u,\kappa_z(V_i)\rangle \\ &= \langle u, V_i \frown z\rangle = \langle u \smile V_i, z\rangle\end{aligned}$$

This equation holds trivially if $\deg u \neq n - i$. The following *Wu formula* shows that the classes V_i and the Stiefel-Whitney classes w_i determine each other.

7 THEOREM *In a compact n-manifold, for $q \geq 0$*

$$w_q = \sum_{0 \leq i \leq q} Sq^{q-i} V_i$$

PROOF We have $U_1 = \sum a_{ij} u_i \times v_j$, where $\{u_i\}$ is a basis of $H^*(X,\mathbf{Z}_2)$ and $\{v_j\}$ is a basis of $H^*(X,\dot{X}; \mathbf{Z}_2)$. By the Cartan formula, condition (d) on page 271

$$Sq^q U_1 = \sum_{k+l=q} a_{ij} Sq^k u_i \times Sq^l v_j$$

Let $V_l = \sum c_{lm} u_m$. Then we have

$$\begin{aligned}w_q = (Sq^q U_1)/z &= \sum_{k+l=q} a_{ij} \langle Sq^l v_j, z\rangle Sq^k u_i \\ &= \sum_{k+l=q} a_{ij} \langle v_j \smile V_l, z\rangle Sq^k u_i \\ &= \sum_{k+l=q} a_{ij} c_{lm} \langle v_j \smile u_m, z\rangle Sq^k u_i \\ &= \sum_{k+l=q} a_{ij} b_{jm} c_{lm} Sq^k u_i\end{aligned}$$

Using lemma 1, we find that

$$w_q = \sum_{k+l=q} c_{li} Sq^k u_i = \sum_{k+l=q} Sq^k V_l \quad \blacksquare$$

Let P^n be the real projective n-space and let w be a generator of $H^1(P^n)$ for any $n \geq 1$. We use lemma 5.9.4 to compute $Sq^i(w^j)$ in the following examples.

8 For the real projective plane P^2, $Sq^1(w) = w^2$; therefore $V_1(P^2) = w$, $w_1(P^2) = w$, and $w_2(P^2) = w^2$.

9 For P^3, $Sq^2(w) = 0$ and $Sq^1(w^2) = 0$, so $V_i(P^3) = 0$ for $i > 0$ and $w_i(P^3) = 0$ for $i > 0$.

10 For P^4, $Sq^2(w^2) = w^4$ and $Sq^1(w^3) = w^4$, so $V_1(P^4) = w$, $V_2(P^4) = w^2$, $w_1(P^4) = w$, $w_2(P^4) = 0$, $w_3(P^4) = 0$, and $w_4(P^4) = w^4$.

11 For P^5, $Sq^2(w^3) = w^5$ and $Sq^1(w^4) = 0$, and $V_2(P^5) = w^2$ is the only nonzero $V_i(P^5)$, where $i > 0$. Hence $w_1(P^5) = 0$, $w_2(P^5) = w^2$, $w_3(P^5) = 0$, $w_4(P^5) = w^4$, and $w_5(P^5) = 0$.

The Euler class and Stiefel-Whitney classes of a manifold X are topological invariants associated to X. We shall now define characteristic classes for a manifold X imbedded in a manifold Y. These will be topological invariants of the imbedding. First, however, we need an algebraic digression.

In our consideration of the slant product we limited ourselves to one of the two possible slant products. We now introduce the other one. Given chain complexes C and C', a cochain $c^* \in \operatorname{Hom}((C \otimes C')_n, G)$, and chain $c \in C_q \otimes G'$, there is a slant product $c\backslash c^* \in \operatorname{Hom}(C'_{n-q}, G \otimes G')$ which is the cochain such that if $c = \sum c_i \otimes g'_i$, with $c_i \in C_q$ and $g'_i \in G'$, then

$$\langle c\backslash c^*, c'\rangle = \sum \langle c^*, c_i \otimes c'\rangle \otimes g'_i \qquad c' \in C'_{n-q}$$

Then
$$\delta(c\backslash c^*) = (-1)^q(c\backslash \delta c^* - \partial c\backslash c^*)$$

from which it follows that there is an induced slant product of $H^n(C \otimes C'; G)$ and $H_q(C;G')$ to $H^{n-q}(C'; G \otimes G')$. This gives rise to a topological slant product of $H^n((X,A) \times (Y,B); G)$ and $H_q(X,A; G')$ to $H^{n-q}(Y,B; G \otimes G')$ having properties analogous to 6.1.1 to 6.1.6. We list without proof two of these, to which we shall have occasion to refer.

12 *Given $u \in H^n((X,A) \times (Y,B); G)$, $z \in H_q(X,A; G'')$, and $v \in H^p(Y,B; G')$, let T: $G \otimes G'' \otimes G' \to G \otimes G' \otimes G''$ interchange the last two factors. In $H^{n-q+p}(Y,B; G \otimes G' \otimes G'')$ we have*

$$T_*((z\backslash u) \smile v) = z\backslash[u \smile (1 \times v)] \quad \blacksquare$$

13 *Given $u \in H^n((X,A) \times (Y,B); G)$, $v \in H^p(X,A; G')$, and $z \in H_q(X,A; G'')$, then, in $H^{n+p-q}(Y,B; G \otimes G' \otimes G'')$,*

$$(v \frown z)\backslash u = z\backslash[u \smile (v \times 1)] \quad \blacksquare$$

Let Y be an m-manifold without boundary and

$$U \in H^m(Y \times Y, Y \times Y - \delta(Y); R)$$

an orientation of Y over R. Given a pair (A,B) in Y, we define

$$\gamma'_U: H_q(A,B; G) \to H^{n-q}(Y - B, Y - A; G)$$

by

$$\gamma'_U(z) = z\backslash[U \mid (A,B) \times (Y - B, Y - A)] \qquad z \in H_q(A,B; G)$$

Then we have the following complement to the duality theorem.

14 LEMMA *Let X be a compact homologically locally connected space in an m-manifold Y with orientation class U. Then we have an isomorphism for all q and all G*

$$\gamma'_U: H_q(X;G) \approx H^{m-q}(Y, Y - X; G)$$

PROOF Since X is compact and homologically locally connected, it follows from theorem 6.9.11 that $H(\Delta(X))$ is of finite type. By lemma 5.5.9, there is a free chain complex C of finite type which is chain equivalent to $\Delta(X)$. Let λ: $C \to \Delta(X)$ be a chain equivalence. Let Δ' and C' be the chain complexes obtained by reindexing the cochain complexes $\operatorname{Hom}(\Delta(X),R)$ and $\operatorname{Hom}(C,R)$, respectively, so that $\Delta'_q = \operatorname{Hom}(\Delta_{m-q}(X),R)$ and $C'_q = \operatorname{Hom}(C_{m-q},R)$. The chain equivalence λ defines a chain equivalence $\lambda'\colon \Delta' \to C'$. Because C is free and of finite type, so is C' [Δ' will not be free, in general, because $\Delta(X)$ need not be of finite type].

Let $c^* \in \operatorname{Hom}([\Delta(X) \otimes (\Delta(Y)/\Delta(Y - X))]_m, R)$ be an m-cocycle corresponding to $U \mid X \times (Y, Y - X)$ under the Eilenberg-Zilber isomorphism and define a map

$$\tau\colon \Delta(Y)/\Delta(Y - X) \to \Delta'$$

by $\tau(c) = c^*/c$ for $c \in \Delta(Y)/\Delta(Y - X)$. If deg $c = q$,

$$\partial(\tau(c)) = \delta(c^*/c) = (-1)^{m-q}c^*/\partial c = (-1)^{m-q}\tau(\partial c)$$

so τ either commutes or anticommutes with ∂, depending on degree. Hence τ induces homomorphisms τ_* on homology and τ^* on cohomology for any coefficient module. Clearly,

$$\tau_* = \gamma_U\colon H_q(Y, Y - X; R) \to H^{m-q}(X;R)$$

Because X is homologically locally connected, by corollary 6.9.8, X is taut in Y, and by the duality theorem, γ_U, and hence τ_*, is an isomorphism. Therefore the composite $\lambda' \circ \tau$ induces an isomorphism $\lambda'_* \circ \tau_*$ of $H_q(Y, Y - X; R)$ with $H_q(C') = H^{m-q}(C;R)$. Since $\Delta(Y)/\Delta(Y - X)$ and C' are both free, it follows from the universal-coefficient formula for cohomology (theorem 5.5.3) that for any G

$$(\lambda' \circ \tau)^* = \tau^* \circ \lambda'^*\colon H^*(\operatorname{Hom}(C',G)) \approx H^*(\operatorname{Hom}(\Delta(Y)/\Delta(Y - X), G))$$

There is also a commutative diagram

$$\begin{array}{ccc}
H_q(\Delta(X) \otimes G) & \xleftarrow[\approx]{(\lambda\otimes 1)_*} & H_q(C \otimes G) \\
\downarrow & & \downarrow\approx \\
H^{m-q}(\operatorname{Hom}(\Delta',G)) & \xleftarrow[\approx]{\lambda'^*} & H^{m-q}(\operatorname{Hom}(C', G))
\end{array}$$

where the vertical maps are induced by the canonical map

$$A \otimes G \to \operatorname{Hom}(\operatorname{Hom}(A,R), G)$$

for any module A (the right-hand vertical map being an isomorphism because C is of finite type). Hence there are isomorphisms

$$H_q(X;G) \xrightarrow{\approx} H^{m-q}(\operatorname{Hom}(\Delta',G)) \xrightarrow[\approx]{\tau^*} H^{m-q}(Y, Y - X; G)$$

It only remains to verify that this composite is γ'_U. If $\sigma \in \Delta_q(X)$ and $g \in G$, the composite

$$\Delta_q(X) \otimes G \to \operatorname{Hom}(\Delta'_{m-q},G) \xrightarrow{\operatorname{Hom}(\tau,1)} \operatorname{Hom}(\Delta_{m-q}(Y)/\Delta_{m-q}(Y - X),G)$$

maps $\sigma \otimes g$ to the homomorphism h such that if $\sigma' \in \Delta_{m-q}(Y)$,

$$\begin{aligned} h(\sigma') &= \tau(\sigma')(\sigma \otimes g) = (c^*/\sigma')(\sigma \otimes g) \\ &= \langle c^*, \sigma \otimes \sigma'\rangle g = [(\sigma \otimes g)\backslash c^*](\sigma') \end{aligned}$$

Therefore $h = (\sigma \otimes g)\backslash c^*$, and this gives the result on passing to homology. ∎

Let X be a closed subset of a space Y tautly imbedded with respect to singular cohomology and let $A \subset X$. Assuming $X - A$ taut in $Y - A$, we define

$$H^p(Y, Y - X; G) \smile H^q(X, X - A; G') \to H^{p+q}(Y, Y - A; G'')$$

where G and G' are paired to G''. If V is any neighborhood of X in Y and V' is a neighborhood of $X - A$ in $V - A$, there is a cup product

$$H^p(V, V - X; G) \smile H^q(V,V'; G') \to H^{p+q}(V, (V - X) \cup V'; G'')$$

There are excision isomorphisms (for all coefficients)

$$H^p(Y, Y - X) \approx H^p(V, V - X)$$
$$H^{p+q}(Y, Y - A) \approx H^{p+q}(V, V - A)$$

Since $V - A = (V - X) \cup V'$ we have a cup product

$$H^p(Y, Y - X; G) \smile H^q(V,V'; G') \to H^{p+q}(Y, Y - A; G'')$$

As V varies over neighborhoods of X in Y and V' varies over neighborhoods of $X - A$ in $V - A$, it follows from the tautness assumptions and the five lemma that $\lim_{\to} \{H^*(V,V'; G')\} \approx H^*(X, X - A; G')$. The desired cup product is thus obtained by passing to the direct limit with the above cup product.

Let X be a compact n-manifold without boundary imbedded in an m-manifold Y without boundary. Assume that U and U' are orientations of X and Y, respectively, over R. There is then an isomorphism (for any R module G)

$$\theta\colon H^q(X;G) \approx H^{m-n+q}(Y, Y - X; G)$$

characterized by commutativity in the triangle of isomorphisms (note that X is homologically locally connected, and so lemma 14 applies to $X \subset Y$)

$$\begin{tikzcd} & H_{n-q}(X;G) \arrow[dl, "\gamma_U"'] \arrow[dr, "\gamma_{U'}"] & \\ H^q(X;G) \arrow[rr, "\theta"] & & H^{m-n+q}(Y, Y - X; G) \end{tikzcd}$$

This map θ is similar to a Thom isomorphism and has the following multiplicative property.

15 LEMMA *The isomorphism $\theta\colon H^q(X,G) \approx H^{m-n+q}(Y, Y - X; G)$ has the property that for $v \in H^q(X;G)$*

$$\theta(v) = \pm\theta(1) \smile v$$

where $\theta(1) \in H^{m-n}(Y, Y - X; R)$

PROOF Let $z \in H_n(X;R)$ be the fundamental class of X corresponding to U and suppose $v = i^*v'$ for $v' \in H^q(V;G)$ and $i\colon X \subset V$, where V is a neighborhood of X in Y. By theorem 6.3.12, $\gamma_U^{-1}(v) = \pm v \frown z = \pm i^*v' \frown z$. Then, using properties 12 and 13 (with all equations holding up to sign),

$$\begin{aligned}
\theta(v) \mid (V, V - X) &= \pm(i^* v' \frown z)\backslash[U' \mid X \times (V, V - X)] \\
&= \pm i_*(i^* v' \frown z)\backslash[U' \mid V \times (V, V - X)] \\
&= \pm(v' \frown i_* z)\backslash[U' \mid V \times (V, V - X)] \\
&= \pm i_* z\backslash\{[U' \mid V \times (V, V - X)] \smile (v' \times 1_V)\} \\
&= \pm i_* z\backslash\{[U' \mid V \times (V, V - X)] \smile (1_V \times v')\} \\
&= \pm z\backslash\{[U' \mid X \times (V, V - X)] \smile (1_X \times v')\} \\
&= \pm[\theta(1) \mid (V, V - X)] \smile v' \\
&= \pm[\theta(1) \smile v] \mid (V, V - X)
\end{aligned}$$

Since $H^*(Y, Y - X) \approx H^*(V, V - X)$, this gives the result. ∎

Our next result, a consequence of lemma 15, follows immediately from the definition of the cup product, $H^p(Y, Y - X) \smile H^q(X) \to H^{p+q}(Y, Y - X)$.

16 COROLLARY *Let X be a compact oriented n-manifold imbedded in an oriented m-manifold Y, both without boundary. For any element $v \in H^q(Y;G)$ we have*

$$\theta(v \mid X) = \pm\theta(1) \smile v \quad ∎$$

The *normal Euler class of X in Y*, denoted by $\chi_{X,Y} \in H^{m-n}(X;R)$, is defined by the equation

$$\theta(\chi_{X,Y}) = \theta(1) \smile \theta(1) \in H^{2(m-n)}(Y, Y - X; R)$$

Since $\theta(1) \smile \theta(1) = \theta(1) \smile [\theta(1) \mid Y]$, we obtain from corollary 16 the following characterization of the normal Euler class.

17 THEOREM *If a compact n-manifold X is imbedded in an m-manifold Y, both without boundary and oriented over R, the normal Euler class $\chi_{X,Y} = \theta(1) \mid X$.* ∎

In particular, if $H^{m-n}(Y;R) \to H^{m-n}(X;R)$ is trivial, it follows that the normal Euler class is zero. Thus, if Y is Euclidean space, the normal Euler class of any compact X imbedded in Y is zero.

For $i \geq 0$ the *ith normal Stiefel-Whitney class of X in Y*, $\bar{w}_i \in H^i(X;\mathbf{Z}_2)$, is defined by

$$\theta(\bar{w}_i) = Sq^i\theta(1)$$

Here are some examples.

18 By condition (a) on page 271, $\bar{w}_0 = 1$.

19 By condition (b) on page 271, if $k = \dim Y - \dim X$ then $\bar{\omega}_k$ is the normal Euler class of X in Y over $\mathbf{Z}_2$.

20 By condition (c) on page 271, $\bar{w}_i = 0$ for $i > \dim Y - \dim X$.

There is an important relation between the Stiefel-Whitney classes of X and Y and the normal Stiefel-Whitney classes of X in Y toward which we are heading.

21 LEMMA *Let X be a compact n-manifold imbedded in an m-manifold Y, both without boundary. Let U and U' be the orientation classes of X and Y, respectively, over $\mathbf{Z}_2$ and let $\theta(1) \in H^{m-n}(Y, Y - X; \mathbf{Z}_2)$. Then*

$$U' \mid (X \times Y, X \times Y - \delta(X)) = [1 \times \theta(1)] \smile U$$

PROOF If X' is a component of X, it suffices to prove that

$$U' \mid (X' \times Y, X' \times Y - \delta(X')) = ([1 \times \theta(1)] \smile U) \mid (X' \times Y, X' \times Y - \delta(X'))$$

Hence we may assume X connected, in which case $(X \times Y, X \times Y - \delta(X))$ is a fiber-bundle pair over X with fiber pair $(Y, Y - x_0)$, where $x_0 \in X$. Since $U' \mid (X \times Y, X \times Y - \delta(X))$ is an orientation over $\mathbf{Z}_2$ of this bundle pair, and there is a unique orientation over $\mathbf{Z}_2$, it suffices to prove that $[1 \times \theta(1)] \smile U$ is also an orientation over $\mathbf{Z}_2$ of this bundle pair. That is, we need only show that for $x \in X$, $([1 \times \theta(1)] \smile U) \mid x \times (Y, Y - x)$ is nonzero. This will be so if its image in $x \times (Y, Y - X)$, which equals $([1 \times \theta(1)] \smile U) \mid x \times (Y, Y - X)$, is nonzero. Because $U \in H^n(X \times X, X \times X - \delta(X))$ is an orientation, $U \mid x \times (X, X - x) = 1_x \times u$, where $u \in H^n(X, X - x)$ is nonzero. Because $H^n(X, X - x) \to H^n(X)$ is a monomorphism [dual to the monomorphism $H_0(x) \to H_0(X)$], $u \mid X$ is nonzero. We have

$$\begin{aligned} ([1 \times \theta(1)] \smile U) \mid x \times (Y, Y - X) &= [1_x \times \theta(1)] \smile (1_x \times u \mid X) \\ &= 1_x \times [\theta(1) \smile u \mid X] = 1_x \times \theta(u \mid X) \end{aligned}$$

Since θ is an isomorphism, this implies that $([1 \times \theta(1)] \smile U) \mid x \times (Y, Y - X)$ is nonzero. ∎

From this result we have the following *Whitney duality theorem.*

22 THEOREM *Let X be a compact n-manifold imbedded in an m-manifold Y, both without boundary. For $k \geq 0$*

$$w_k(Y) \mid X = \sum_{i+j=k} \bar{w}_i \smile w_j(X)$$

where $w_k(Y)$, $w_j(X)$, and $\bar{w}_i$ denote the Stiefel-Whitney classes of Y, X, and X in Y, respectively.

PROOF The result follows easily from lemma 21 and the Cartan formula (rather, the equivalent form of lemma 5.9.4):

$$\begin{aligned} &([w_k(Y) \mid X] \times 1_Y) \smile U' \mid (X \times Y, X \times Y - \delta(X)) \\ &\quad = ([w_k(Y) \times 1_Y] \smile U') \mid (X \times Y, X \times Y - \delta(X)) \\ &\quad = Sq^k U' \mid (X \times Y, X \times Y - \delta(X)) = Sq^k(U' \mid (X \times Y, X \times Y - \delta(X))) \\ &\quad = Sq^k([1_X \times \theta(1)] \smile U) = \sum_{i+j=k} [1_X \times Sq^i\theta(1)] \smile Sq^j U \\ &\quad = \sum_{i+j=k} (1_X \times [\theta(1) \smile \bar{w}_i]) \smile [w_j(X) \times 1_X] \smile U \\ &\quad = \sum_{i+j=k} (\bar{w}_i \times 1_X) \smile [w_j(X) \times 1_X] \smile [1_X \times \theta(1)] \smile U \\ &\quad = (([\sum_{i+j=k} \bar{w}_i \smile w_j(X)] \times 1_Y) \smile U') \mid (X \times Y, X \times Y - \delta(X)) \end{aligned}$$

By the Thom isomorphism theorem, this implies the result. ■

In case Y is Euclidean space, $w_k(Y) = 0$ for $k > 0$, and theorem 22 shows that $\bar{w}_i$ and $w_j(X)$ determine each other recursively. In particular, the classes $\bar{w}_i$ are independent of the imbedding of X in the Euclidean space. If X is a compact n-manifold imbedded in $\mathbf{R}^{n+d}$, it follows from example 19 and 20 and from the fact that the Euler class of X in $\mathbf{R}^{n+d}$ is zero that $\bar{w}_i = 0$ for $i \geq d$. This gives the following necessary condition for imbeddability of X in $\mathbf{R}^{n+d}$.

23 COROLLARY *Let X be a compact n-manifold imbedded in $\mathbf{R}^{n+d}$ and let $\bar{w}_i \in H^i(X;\mathbf{Z}_2)$ be defined by*

$$\sum_{i+j=k} \bar{w}_i \smile w_j(X) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

Then $\bar{w}_i = 0$ for $i \geq d$. ■

We present some examples.

24 For P^2, $\bar{w}_1(P^2) = w$ and $\bar{w}_2(P^2) = 0$, so P^2 cannot be imbedded in $\mathbf{R}^3$.

25 For P^3, $\bar{w}_i(P^3) = 0$ for $i > 0$.

26 For P^4, $\bar{w}_1(P^4) = w$, $\bar{w}_2(P^4) = w^2$, $\bar{w}_3(P^4) = w^3$, and $\bar{w}_4(P^4) = 0$. Therefore P^4 cannot be imbedded in $\mathbf{R}^7$.

27 For P^5, $\bar{w}_1(P^5) = 0$, $\bar{w}_2(P^5) = w^2$, $\bar{w}_3(P^5) = 0$, $\bar{w}_4(P^5) = 0$, and $\bar{w}_5(P^5) = 0$. Hence P^5 cannot be imbedded in $\mathbf{R}^7$ (which is also a consequence of example 26).

The last examples show the importance of calculating $w_i(P^n)$, which we now do.

28 THEOREM *Let $\binom{n}{i}_2$ be the binomial coefficient $\binom{n}{i} = n!/i!(n-i)!$ reduced modulo 2. Then*

$$w_i(P^n) = \binom{n+1}{i}_2 w^i$$

PROOF Since $\binom{n+1}{n}_2 \equiv n + 1 = \chi(P^n)$, the result is true for $i = n$. For $i < n$, where $n > 1$, we suppose P^{n-1} linearly imbedded in P^n. Then $P^n - P^{n-1}$ is an affine space, hence $\tilde{H}^*(P^n - P^{n-1}) = 0$ and $H^q(P^n, P^n - P^{n-1}) \approx \tilde{H}^q(P^n)$. Then the normal Thom class $\theta(1) \in H^1(P^n, P^n - P^{n-1})$ maps to w in $H^q(P^n)$, so $\bar{w}_1 = w$. By theorem 22, $w_i(P^n) \mid P^{n-1} = w_i(P^{n-1}) + w \smile w_{i-1}(P^{n-1})$. Since $H^q(P^n) \approx H^q(P^{n-1})$ for $q < n$, it follows by induction on n that

$$w_i(P^n) = [\binom{n}{i-1}_2 + \binom{n}{i}_2]w^i = \binom{n+1}{i}_2 w^i \quad ■$$

EXERCISES

A MANIFOLDS

1 If X is an n-manifold with boundary $\dot{X}$, prove that X is a homology n-manifold whose boundary, as a homology manifold, equals $\dot{X}$.

In the rest of the exercises of this group, X will be an n-manifold without boundary and R will be a fixed principal ideal domain.

2 If Γ is a local system of R modules on X, prove that for any $A \subset X$

$$H_q(A \times X, A \times X - \delta(A); R \times \Gamma) = 0 \qquad q < n$$

(*Hint:* Prove this first for $\bar{A}$ contained in a coordinate neighborhood of X. Prove it next for compact $\bar{A}$ by using the Mayer-Vietoris technique. Then prove it for arbitrary A by taking direct limits over the family of compact subsets of A.)

3 Prove that there is a local system Γ_X of R modules on X such that $\Gamma_X(x) = H^n(X, X - x; R)$ for $x \in X$.

For $x \in X$ let $z_x \in H_n(X, X - x; \Gamma_X)$ be the generator corresponding under the isomorphism

$$H_n(X, X - x; \Gamma_X) \approx \text{Hom}\,(H^n(X, X - x; R), H^n(X, X - x; R))$$

to the identity homomorphism of $H^n(X, X - x; R)$. A *Thom class* of X is an element

$$U \in H^n(X \times X, X \times X - \delta(X); R \times \text{Hom}\,(\Gamma_X,R))$$

such that $(U \mid [x \times (X, X - x)])/z_x = 1 \in H^0(x;R)$ for all $x \in X$.

4 If V is an open subset of X and U is a Thom class of X, prove that $U \mid (V \times V, V \times V - \delta(V))$ is a Thom class of V.

5 Prove that $\mathbf{R}^n$ has a unique Thom class.

6 Prove that X has a unique Thom class. [*Hint:* Use exercise 2 to show that

$$H^n(X \times X, X \times X - \delta(X); R \times \text{Hom}\,(\Gamma_X,R)) \approx \lim_{\leftarrow} \{H^n(V \times X, V \times X - \delta(V); R \times \text{Hom}\,(\Gamma_X,R))\}$$

where V varies over finite unions of coordinate neighborhoods. Then the result follows from exercises 4 and 5 by Mayer-Vietoris techniques.]

If (A,B) is a pair in X and G is an R module, define

$$\gamma\colon H_q(X - B, X - A; \Gamma_X \otimes G) \to H^{n-q}(A,B; G)$$

by $\gamma(z) = [U \mid (A,B) \times (X - B, X - A)]/z$, where U is the Thom class of X. As (V,W) varies over neighborhoods of a closed pair (A,B) in X, there are isomorphisms

$$\lim_{\to} \{H_q(X - W, X - V; \Gamma_X \otimes G)\} \approx H_q(X - B, X - A; \Gamma_X \otimes G)$$

and

$$\lim_{\to} \{H^{n-q}(V,W; G)\} \approx \bar{H}^{n-q}(A,B; G)$$

and a homomorphism

$$\bar{\gamma}\colon H_q(X - B, X - A; \Gamma_X \otimes G) \to \bar{H}^{n-q}(A,B; G)$$

is defined by passing to the limit with γ.

7 *Duality theorem.* Prove that for a compact pair (A,B) in X, $\bar{\gamma}$ is an isomorphism.

B THE INDEX OF A MANIFOLD

1 Let X be a compact n-manifold, with boundary $\dot{X}$ oriented over a field R, and let $[X] \in H_n(X,\dot{X}; R)$ be the corresponding fundamental class. For $u \in H^q(X,\dot{X}; R)$ and $v \in H^{n-q}(X;R)$ prove that $\varphi_X(u,v) = \langle u \smile v, [X]\rangle \in R$ is a nonsingular bilinear form from $H^q(X,\dot{X}) \times H^{n-q}(X)$ to R [that is, $u = 0$ if and only if $\varphi_X(u,v) = 0$ for all v].

2 With the same hypotheses as above, let $[\dot{X}] = \partial[X] \in H_{n-1}(\dot{X};R)$ and let $\varphi_{\dot{X}}$ be the corresponding bilinear form from $H^{q-1}(\dot{X};R) \times H^{n-q}(\dot{X};R)$ to R. Let $j\colon \dot{X} \subset X$, and if $u \in H^{q-1}(\dot{X};R)$ and $v \in H^{n-q}(X;R)$, prove that

$$\varphi_{\dot{X}}(u,j^*(v)) = \varphi_X(\delta(u),v)$$

3 Prove that the Euler characteristic of any odd-dimensional compact manifold is 0 and the Euler characteristic of an even-dimensional compact manifold which is a boundary is even. (*Hint:* If $\dot{X}$ is the boundary of a $(2n+1)$-manifold X, then, with $\mathbf{Z}_2$ coefficients,

$$\dim \operatorname{im} [j^*: H^n(X) \to H^n(\dot{X})] = \dim \operatorname{im} [\delta: H^n(\dot{X}) \to H^{n+1}(X,\dot{X})]$$

and their sum equals $\dim H^n(\dot{X})$.)

Let Y be a compact $4m$-manifold, without boundary oriented over $\mathbf{R}$, and define the *index* of Y to be the index of the nonsingular bilinear form φ_Y from $H^{2m}(Y;\mathbf{R}) \times H^{2m}(Y;\mathbf{R})$ to $\mathbf{R}$ (when φ_Y is represented as a sum of k squares minus a sum of j squares, the index of φ_Y is $k - j$).

4 If Y is oppositely oriented, prove that its index changes sign. Show that the index of the product of oriented manifolds is the product of their indices.

5 If X is a compact $(4m+1)$-manifold, with boundary $\dot{X}$ oriented over $\mathbf{R}$, prove that The index of $\dot{X}$ is 0. [*Hint:* Prove that $j^*(H^{2m}(X;\mathbf{R}))$ is a subspace of $H^{2m}(\dot{X};\mathbf{R})$ whose dimension equals one-half the dimension of $H^{2m}(\dot{X};\mathbf{R})$ and on which $\varphi_{\dot{X}}$ is identically zero. This implies the result.]

C CONTINUITY

1 Let $\{(X_j,A_j), \pi_j{}^k\}_{j\in J}$ be an inverse system of compact Hausdorff pairs and let $(X,A) = \lim_{\leftarrow} \{(X_j,A_j)\}$. Prove that (X,A) can be imbedded in a space in which it is a directed intersection of compact Hausdorff pairs $\{(X_j',A_j')\}_{j\in J}$, where (X_j',A_j') has the same homotopy type as (X_j,A_j). [*Hint:* For each $j \in J$ imbed X_j in a contractible compact Hausdorff space Y_j, for example, a cube, and let $(X_k',A_k') \subset \times_{j\in J} Y_j$ be defined as the pair of all points (y_j) with y_k in X_k or in A_k, respectively, such that if $j \le k$, then $y_j = \pi_j{}^k(y_k)$, and if $j \not\le k$, then y_j is arbitrary.]

2 Prove that a cohomology theory has the continuity property if and only if it has the weak continuity property.

3 The *p-adic solenoid* is defined to be the inverse limit of the sequence

$$S^1 \xleftarrow{f} S^1 \leftarrow \cdots \leftarrow S^1 \xleftarrow{f} S^1 \leftarrow \cdots$$

where $f(z) = z^p$. Compute the Alexander cohomology groups of the p-adic solenoid for coefficients $\mathbf{Z}$, $\mathbf{Z}_p$, and $\mathbf{R}$.

4 Generalize the solenoid of the preceding example to the case where there is a sequence of integers $n_1, n_2, \ldots$ such that the mth map of S^1 to S^1 sends z to z^{n_m}. Compute the integral Alexander cohomology groups of the resulting space.

5 Find a compact Hausdorff space X such that $\bar{\bar{H}}^q(X;\mathbf{Z}) = 0$ if $q \ne 1$ and $\bar{H}^1(X;\mathbf{Z}) \approx \mathbf{R}$.

D ČECH COHOMOLOGY THEORY

1 Let $(\mathfrak{U},\mathfrak{U}')$ be an open covering of (X,A) ($\mathfrak{U}$ is an open covering of X and $\mathfrak{U}' \subset \mathfrak{U}$ is a covering of A) and let $K(\mathfrak{U})$ be the nerve of $\mathfrak{U}$ and $K'(\mathfrak{U}')$ the subcomplex of $K(\mathfrak{U})$ which is the nerve of $\mathfrak{U}' \cap A = \{U' \cap A \mid U' \in \mathfrak{U}'\}$. Prove that the chain complexes $(C(K(\mathfrak{U})),C(K'(\mathfrak{U}')))$ and $(C(X(\mathfrak{U})),C(A(\mathfrak{U}')))$ are canonically chain equivalent. (*Hint:* If $s = \{U_0, \ldots ,U_q\}$ is a simplex of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$], let $\lambda(s)$ be the subcomplex of $X(\mathfrak{U})$ [or of $A(\mathfrak{U}')$] generated by all simplexes of $X(\mathfrak{U})$ [or of $A(\mathfrak{U}')$] in $\bigcap U_i$. If $s' = \{x_0, \ldots ,x_q\}$ is a simplex of $X(\mathfrak{U})$ [or of $A'(\mathfrak{U}')$], let $\mu(s')$ be the subcomplex of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$] generated by all simplexes $\{U_0, \ldots, U_r\}$ of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$] such that U_i contains s' for $0 \le i \le r$. Then $C(\lambda(s))$ and $C(\mu(s'))$ are acyclic, and the method of acyclic models can be applied to prove the existence of chain maps

$$\tau\colon (C(K(\mathfrak{U})),C(K'(\mathfrak{U}'))) \to (C(X(\mathfrak{U})),C(A(\mathfrak{U}')))$$
$$\tau'\colon (C(X(\mathfrak{U})),C(A(\mathfrak{U}))) \to (C(K(\mathfrak{U})),C(K'(\mathfrak{U}')))$$

such that $\tau(C(s)) \subset C(\lambda(s))$ and $\tau'(C(s')) \subset C(\mu(s'))$. Similarly, the method of acyclic models shows that τ and τ' are chain homotopy inverses of each other.[1])

2 Let $(\mathfrak{V},\mathfrak{V}')$ be a refinement of $(\mathfrak{U},\mathfrak{U}')$, let $\pi\colon (K(\mathfrak{V}),K'(\mathfrak{V}')) \to (K(\mathfrak{U}),K'(\mathfrak{U}'))$ be a projection map, and let $j\colon (X(\mathfrak{V}),A(\mathfrak{V}')) \subset (X(\mathfrak{U}),A(\mathfrak{U}'))$. For any abelian group G prove that there is a commutative diagram

$$\begin{array}{ccc} H^*(K(\mathfrak{U}),K'(\mathfrak{U}');G) & \approx & H^*(X(\mathfrak{U}),A(\mathfrak{U}');G) \\ \pi^*\downarrow & & \downarrow j^* \\ H^*(K(\mathfrak{V}),K'(\mathfrak{V}');G) & \approx & H^*(X(\mathfrak{V}),A(\mathfrak{V}');G) \end{array}$$

where the horizontal maps are induced by the canonical chain equivalences of exercise 1 above.

3 The *Čech cohomology group* of (X,A) with coefficients G is defined by $\check{H}^*(X,A;G) = \lim_{\rightarrow} \{H^*(K(\mathfrak{U}),K'(\mathfrak{U}');G)\}$. Prove that there is a natural isomorphism

$$\check{H}^*(X,A;G) \approx \bar{H}^*(X,A;G).$$

4 If $\dim (X - A) \le n$, prove that $\bar{H}^q(X,A;G) = 0$ for all $q > n$ and all G.

E THE KÜNNETH FORMULA FOR ČECH COHOMOLOGY

If K_1 and K_2 are simplicial complexes, their *simplicial product* $K_1 \Delta K_2$ is the simplicial complex whose vertex set is the cartesian product of the vertex sets of K_1 and of K_2 and whose simplexes are sets $\{(v_0,w_0), \ldots ,(v_q,w_q)\}$, where $v_0, \ldots , v_q$ are vertices of some simplex of K_1 and $w_0, \ldots , w_q$ are vertices of some simplex of K_2.

1 Prove that $K_1 \Delta K_2$ is a simplicial complex, and if $L_1 \subset K_1$ and $L_2 \subset K_2$, then $L_1 \Delta L_2 \subset K_1 \Delta K_2$.

2 For simplicial pairs (K_1,L_1) and $K_2,L_2)$ define

$$(K_1,L_1) \Delta (K_2,L_2) = (K_1 \Delta K_2, K_1 \Delta L_2 \cup L_1 \Delta K_2)$$

Prove that $C((K_1,L_1) \Delta (K_2,L_2))$ is canonically chain equivalent to $C(K_1,L_1) \otimes C(K_2,L_2)$. (*Hint:* Use the method of acyclic models.)

3 Call an open covering $(\mathcal{U},\mathcal{U}')$ of (X,A) *special* if $\mathcal{U}' = \{U \in \mathcal{U} \mid U \cap A \neq \phi\}$. If $(\mathcal{U},\mathcal{U}')$ is a special open covering of (X,A) and $(\mathcal{V},\mathcal{V}')$ is a special open covering of (Y,B), let $(\mathcal{U},\mathcal{U}') \times (\mathcal{V},\mathcal{V}') = (\mathcal{W},\mathcal{W}')$ be the special open covering of $(X,A) \times (Y,B)$ where $\mathcal{W} = \{U \times V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ and $\mathcal{W}' = \{U \times V \mid U \in \mathcal{U}' \text{ or } V \in \mathcal{V}'\}$. Prove that $(K(\mathcal{W}), K'(\mathcal{W}')) = (K(\mathcal{U}), K'(\mathcal{U}')) \Delta (K(\mathcal{V}), K'(\mathcal{V}'))$.

4 If A is closed in X, prove that the family of special open coverings of (X,A) is cofinal in the family of all open coverings of (X,A). If (X,A) and (Y,B) are compact Hausdorff pairs, prove that the family of coverings of $(X,A) \times (Y,B)$ of the form $(\mathcal{U},\mathcal{U}') \times (\mathcal{V},\mathcal{V}')$ where $(\mathcal{U},\mathcal{U}')$ is a special open covering of (X,A) and $(\mathcal{V},\mathcal{V}')$ is a special open covering of (Y,B) is cofinal in the family of all open coverings of $(X,A) \times (Y,B)$.

[1] For details see C. H. Dowker, Homology groups of relations, *Annals of Mathematics*, vol. 56, pp. 84–95, 1952.

5 If (X,A) and (Y,B) are compact Hausdorff pairs and G and G' are modules such that $G * G' = 0$, prove that there is a short exact sequence

$$0 \to (\check{H}_1^* \otimes \check{H}_2^*)^q \to \check{H}^q((X,A) \times (Y,B); G \otimes G') \to (\check{H}_1^* * \check{H}_2^*)^{q+1} \to 0$$

where $\check{H}_1^* = \check{H}^*(X,A; G)$ and $\check{H}_2^* = \check{H}^*(Y,B; G')$.

6 Let (X,A) and (Y,B) be locally compact Hausdorff pairs with A and B closed in X and Y, respectively. If G and G' are modules such that $G * G' = 0$, prove that there is a short exact sequence

$$0 \to (\bar{H}_{c,1}^* \otimes \bar{H}_{c,2}^*)^q \to \bar{H}_c{}^q((X,A) \times (Y,B); G * G') \to (\bar{H}_{c,1}^* * \bar{H}_{c,2}^*)^{q+1} \to 0$$

where $\bar{H}_{c,1}^* = \bar{H}_c^*(X,A; G)$ and $\bar{H}_{c,2}^* = \bar{H}_c^*(Y,B; G')$.

F LOCAL SYSTEMS AND SHEAVES

Throughout this group of exercises we assume X to be a paracompact Hausdorff space.

1 If Γ is a local system on X, let $\bar{\Gamma}$ be the presheaf on X such that for an open set $V \subset X$, $\bar{\Gamma}(V)$ is the set of all functions f assigning to each $x \in X$ an element $f(x) \in \Gamma(x)$ with the property that for any path ω in V, $f(\omega(1)) = \Gamma(\omega)(f(\omega(0)))$. Prove that $\bar{\Gamma}$ is a sheaf on X and the association of $\bar{\Gamma}$ to Γ is a natural transformation from local systems to sheaves.

2 A presheaf Γ on X is said to be *locally constant* if there is an open covering $\mathfrak{U} = \{U\}$ of X such that if $U \in \mathfrak{U}$ and $x \in U$, then $\Gamma(U) \approx \lim_{\to} \{\Gamma(V)\}$, where V varies over open neighborhoods of x. If $U \in \mathfrak{U}$ and U' is a connected open subset of U, prove that the composite

$$\Gamma(U) \to \Gamma(U') \to \hat{\Gamma}(U')$$

is an isomorphism. Deduce that if Γ is a locally constant sheaf and U' is a connected open subset of $U \in \mathfrak{U}$, then $\Gamma(U) \approx \Gamma(U')$.

3 If X is locally path connected and Γ' is a locally constant sheaf on X, prove that there is a local system Γ on X such that $\bar{\Gamma} \approx \Gamma'$.

4 If X is locally path connected and semilocally 1-connected, prove that there is a one-to-one correspondence between equivalence classes of local systems on X and equivalence classes of locally constant sheaves on X.

5 If Γ is a local system of R modules on X, let $\Delta^q(\cdot\,;\Gamma)$ be the presheaf on X such that $\Delta^q(\cdot\,;\Gamma)(V) = \Delta^q(V;\Gamma \mid V)$ for V open in X. Prove that $\Delta^q(\cdot\,;\Gamma)$ is fine.

6 If Γ is a local system of R modules on X, let $\Delta^*(\cdot\,;\Gamma)$ be the cochain complex of presheaves $\Delta^q(\cdot\,;\Gamma)$ on X and let $\hat{\Delta}^*(\cdot\,;\Gamma)$ be the cochain complex of completions $\hat{\Delta}^q(\cdot\,;\Gamma)$. Prove that there is an isomorphism

$$H^*(\Delta^*(\cdot\,;\Gamma)(X)) \approx H^*(\hat{\Delta}^*(\cdot\,;\Gamma)(X))$$

7 Let Γ be a local system of R modules on X and assume that $H^q(\Delta^*(\cdot\,;\Gamma))$ is locally zero on X for all $q > 0$. Prove that there is an isomorphism

$$\check{H}^*(X;\bar{\Gamma}) \approx H^*(X;\Gamma)$$

(*Hint:* Note that $\bar{\Gamma} = H^0(\Delta^*(\cdot\,;\Gamma))$ and apply theorem 6.8.7.)

G SOME PROPERTIES OF EUCLIDEAN SPACE

1 Find a compact subset X of $\mathbf{R}^2$ that is n-connected for all n and such that $\bar{H}^1(X;\mathbf{Z}) \approx \mathbf{Z}$.

2 If X is a compact subset of $\mathbf{R}^n$ and dim $X < n - 1$, prove that $\mathbf{R}^n - X$ is connected.

Let A_1 and A_2 be disjoint closed subsets of $\mathbf{R}^n$ and let $z_1 \in H_p(A_1;R)$ and $z_2 \in H_q(A_2;R)$, with $p + q = n - 1$. If $z_1 \in \tilde{H}_p(A_1;R)$, let $z_1' \in H_{p+1}(\mathbf{R}^n,\mathbf{R}^n - A_2;R)$ be the image of z_1 under the composite

$$\tilde{H}_p(A_1) \to \tilde{H}_p(\mathbf{R}^n - A_2) \xrightarrow[\approx]{\partial^{-1}} H_{p+1}(\mathbf{R}^n, \mathbf{R}^n - A_2)$$

The *linking number* Lk $(z_1,z_2) \in R$ is defined by

$$\text{Lk } (z_1,z_2) = \langle \gamma_U(z_1'),z_2 \rangle$$

where U is an orientation class of $\mathbf{R}^n$ over R fixed once and for all.

3 Prove that Lk $(z_1,z_2) = \langle U, i_*(z_2 \times z_1') \rangle$, where

$$i\colon A_2 \times (\mathbf{R}^n, \mathbf{R}^n - A_2) \subset (\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n - \delta(\mathbf{R}^n))$$

4 Assume that Lk (z_2,z_1) is also defined [that is, $z_2 \in \tilde{H}_q(A_2)$]. Prove that Lk $(z_1,z_2) = (-1)^{pq+1}$ Lk (z_2,z_1).

5 Let A_1 be a p-sphere and A_2 a q-sphere imbedded as disjoint subsets of $\mathbf{R}^n$, where $p + q = n - 1$. Prove that $H_p(A_1) \to H_p(\mathbf{R}^n - A_2)$ is trivial if and only if $H_q(A_2) \to H_q(\mathbf{R}^n - A_1)$ is trivial.

H IMBEDDINGS OF MANIFOLDS IN EUCLIDEAN SPACE

1 Prove that a compact n-manifold which is nonorientable over $\mathbf{Z}$ cannot be imbedded in $\mathbf{R}^{n+1}$.

2 Let X be a compact connected n-manifold imbedded in $\mathbf{R}^{n+1}$ and let U and V be the components of $\mathbf{R}^{n+1} - X$. Let $i\colon X \subset \mathbf{R}^{n+1} - U$ and $j\colon X \subset \mathbf{R}^{n+1} - V$ and prove that over any R, $i^*(\bar{H}^*(\mathbf{R}^{n+1} - U))$ and $j^*(\bar{H}^*(\mathbf{R}^{n+1} - V))$ are subalgebras of $\bar{H}^*(X)$

$$\{i^*,j^*\}\colon \bar{H}^q(\mathbf{R}^{n+1} - U) \oplus \bar{H}^q(\mathbf{R}^{n+1} - V) \approx \bar{H}^q(X) \qquad 0 < q < n$$

3 Prove that for $n \geq 2$ the real projective n-space P^n cannot be imbedded in $\mathbf{R}^{n+1}$.

CHAPTER SEVEN
HOMOTOPY THEORY

WITH THIS CHAPTER WE RETURN TO THE CONSIDERATION OF GENERAL HOMOTOPY theory. Now that we have homology theory available as a tool, we are able to obtain deeper results about homotopy than we could without it. We shall consider the higher homotopy groups in some detail and prove they satisfy analogues of all the axioms of homology theory except the excision axiom. We introduce the Hurewicz homomorphism as a natural transformation from the homotopy groups to the integral singular homology groups. It leads us to the Hurewicz isomorphism theorem, which states roughly that the lowest-dimensional nontrivial homotopy group is isomorphic to the corresponding integral homology group.

We discuss next the concept of CW complex. The class of CW complexes is particularly suited for homotopy theory because it is the smallest class of spaces containing the empty space and, up to homotopy type, is closed with respect to the operation of attaching cells (even an infinite number).

The last main result is the Brown representability theorem. It characterizes by means of simple properties those contravariant functors from the homotopy category of path-connected pointed CW complexes to the category of pointed sets that are naturally equivalent to the functor assigning to a CW complex the set of homotopy classes of maps from it to some fixed pointed space.

Section 7.1 contains a general exactness property for sets of homotopy classes. Section 7.2 contains definitions of the absolute and relative homotopy groups and proofs of the exactness of the homotopy sequences of a pair, a triple, and a fibration. In Sec. 7.3 we consider the extent to which the homotopy groups depend on the choice of the base point used in their definition and prove analogues for the higher homotopy groups of properties established in Chapter One for the fundamental group.

The Hurewicz homomorphism is defined in Sec. 7.4 and the Hurewicz isomorphism theorem is proved in Sec. 7.5. The proof establishes the absolute and relative Hurewicz theorems, as well as a homotopy addition theorem, by simultaneous induction. The Hurewicz theorem implies the Whitehead theorem, which asserts that a continuous map between simply connected spaces induces isomorphisms of all homotopy groups if and only if it induces isomorphisms of all integral singular homology groups.

Section 7.6 introduces the concept of CW complex. Among the elementary properties established is the cellular-approximation theorem, which is an analogue for CW complexes of the simplicial-approximation theorem. Section 7.7 deals with contravariant functors on the homotopy category of path-connected pointed spaces. We prove the representability theorem cited above, and apply it in Sec. 7.8 to obtain CW approximations to a space or a pair and to discuss the related concept of weak homotopy type. The representability theorem will be used again in Chapter Eight.

1 EXACT SEQUENCES OF SETS OF HOMOTOPY CLASSES

One of the most important properties of the homology functor is the exactness property relating the homology of the pair and the homology of each of the spaces in the pair. A similar exactness property is valid for functors defined by homotopy classes. This section is devoted to preliminaries about homotopy classes and a proof of this exactness property. Throughout the section we shall work in the category of pointed spaces, and unless stated to the contrary, (X,A) will be understood as a pair of pointed spaces (that is, A has the same base point as X) in which the subspace A and the base point are closed in X. Homotopies in this category are understood to preserve base points. If $A \subset X$, we use X/A to denote the space obtained from X by collapsing A to a single point (this point serving as the base point of X/A). If X' and A are closed subsets of X, then $A/(A \cap X')$ is a closed subset of X/X'. Hence, if (X,A) is a pair and X' is closed in X, there is a pair $(X/X', A/(A \cap X'))$, which will also be denoted by $(X,A)/X'$.

The unit interval I will be a pointed space with 0 as base point. The *reduced cone* CX over X is defined to be the space obtained from $X \times I$ by collapsing $X \times 0 \cup x_0 \times I$ to a point (so $CX = X \times I/(X \times 0 \cup x_0 \times I)$). We shall use $[x,t]$ to denote the point of CX corresponding to the point $(x,t) \in X \times I$ under the collapsing map $X \times I \to CX$. X is imbedded as a closed subset of CX by the map $x \to [x,1]$. If (X,A) is a pair, then CA is a subspace of CX and $C(X,A)$ is defined to be the pair (CX,CA).

1 LEMMA *A map f: $(X,A) \to (Y,B)$ is null homotopic if and only if there is a map F: $C(X,A) \to (Y,B)$ such that $F[x,1] = f(x)$ for all $x \in X$.*

PROOF There is a one-to-one correspondence between null homotopies H: $(X,A) \times I \to (Y,B)$ of f and maps F:$C(X,A) \to (Y,B)$ such that $F[x,1] = f(x)$, given by the formula

$$F[x,t] = H(x, 1 - t) \quad \blacksquare$$

The following relative homotopy extension property can also be deduced from the relative form of theorem 1.4.12.

2 LEMMA *Given f: $C(X,A) \to (Y,B)$ and a homotopy G: $(X,A) \times I \to (Y,B)$ of $f \mid (X,A)$, there is a homotopy F: $C(X,A) \times I \to (Y,B)$ of f such that $F \mid (X,A) \times I = G$.*

PROOF An explicit formula for F is

$$F([x,t], t') = \begin{cases} f[x, t(1 + t')] & t(1 + t') \leq 1 \\ G(x, t(1 + t') - 1) & 1 \leq t(1 + t') \end{cases} \quad \blacksquare$$

The homotopy class of the unique constant map $(X,A) \to (Y,B)$ is denoted by $0 \in [X,A; Y,B]$ [it consists of the null-homotopic maps $(X,A) \to (Y,B)$]. Because the composite, on either side, of a null-homotopic map and an arbitrary map is null homotopic, the element 0 is a distinguished element of $[X,A; Y,B]$, and we regard $[X,A; Y,B]$ as a pointed set with this distinguished element. Given a map f: $(X',A') \to (X,A)$, the *kernel* of the induced map

$$f^{\#}: [X,A; Y,B] \to [X',A'; Y,B]$$

is defined to be the pointed set $f^{\#-1}(0)$ and is denoted by $\ker f^{\#}$.

We now show how to map another set of homotopy classes into $[X,A; Y,B]$ so that its image equals $\ker f^{\#}$. This will be the basis for the exactness property we seek. The *mapping cone* C_f of a map f: $X' \to X$ is defined to be the quotient space of $CX' \vee X$ by the identifications $[x',1] = f(x')$ for all $x' \in X'$. Given a map f: $(X',A') \to (X,A)$, let f': $X' \to X$ and f'': $A' \to A$ be maps defined by f. Then $C_{f''}$ is a closed subspace of $C_{f'}$ and there is a pair $(C_{f'},C_{f''})$. There is a functorial imbedding i of (X,A) as a closed subpair of $(C_{f'},C_{f''})$.

A three-term sequence of pairs and maps

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{g} (X'',A'')$$

is said to be *exact* if for any pair (Y,B) (where B is not necessarily closed in Y) the associated sequence of pointed sets

$$[Y,B;\ X',A'] \xrightarrow{f_\#} [Y,B;\ X,A] \xrightarrow{g_\#} [Y,B;\ X'',A'']$$

is exact (that is, $\ker g_\# = \operatorname{im} f_\#$). Similarly, it is said to be *coexact* if the sequence of pointed sets

$$[X'',A'';\ Y,B] \xrightarrow{g^\#} [X,A;\ Y,B] \xrightarrow{f^\#} [X',A';\ Y,B]$$

is exact (that is, $\ker f^\# = \operatorname{im} g^\#$). A sequence of pairs and maps (which may terminate at either or both ends)

$$\cdots \to (X_{n+1},A_{n+1}) \xrightarrow{f_n} (X_n,A_n) \xrightarrow{f_{n-1}} (X_{n-1},A_{n-1}) \to \cdots$$

is said to be an *exact sequence* (or a *coexact sequence*) if every three-term sequence of consecutive pairs is exact (or coexact).

3 THEOREM *For any map f: $(X',A') \to (X,A)$ the sequence*

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''})$$

is coexact.

PROOF Let (Y,B) be arbitrary (with B not necessarily closed in Y) and consider the sequence

$$[C_{f'},C_{f''};\ Y,B] \xrightarrow{i^\#} [X,A;\ Y,B] \xrightarrow{f^\#} [X',A';\ Y,B]$$

We now show that $\operatorname{im} i^\# \subset \ker f^\#$. The composite $i \circ f$: $(X',A') \to (C_{f'},C_{f''})$ equals the composite

$$(X',A') \subset C(X',A') \subset C(X',A') \vee (X,A) \xrightarrow{k} (C_{f'},C_{f''})$$

where k is the canonical map to the quotient. However, the inclusion map $(X',A') \subset C(X',A')$ is null homotopic [by lemma 1, because this inclusion map can be extended to the identity map of $C(X',A')$]. Therefore $i \circ f$ is null homotopic, and so $\operatorname{im}(f^\# \circ i^\#) = 0$, proving that $\operatorname{im} i^\# \subset \ker f^\#$.

Assume that g: $(X,A) \to (Y,B)$ is such that $f^\#[g] = 0$ (that is, $g \circ f$ is null homotopic). By lemma 1, there is a map G: $C(X',A') \to (Y,B)$ which extends $g \circ f$. Then G and g define a map G': $C(X',A') \vee (X,A) \to (Y,B)$ such that $G' \mid C(X',A') = G$ and $G' \mid (X,A) = g$. Since

$$G'[x',1] = G[x',1] = g(f(x')) = G'(f(x')) \qquad x' \in X'$$

there is a map h: $(C_{f'},C_{f''}) \to (Y,B)$ such that $G' = h \circ k$. Then $h \mid (X,A) = g$, showing that $h \circ i = g$ or $[g] = i^\#[h]$. Therefore $\ker f^\# \subset \operatorname{im} i^\#$. ∎

For a map f: $(X',A') \to (X,A)$ we have a sequence

4 $$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{j} (C_{i'},C_{i''}) \xrightarrow{l} (C_{j'},C_{j''})$$

and by theorem 3, it follows that this sequence is coexact.

Thus we have succeeded in imbedding the map

$$f^\#: [X,A;\ Y,B] \to [X',A';\ Y,B]$$

in an exact sequence. We shall show that the pairs $(C_{i'},C_{i''})$ and $(C_{j'},C_{j''})$ in sequence 4 can be replaced by other pairs more explicitly expressed in terms of (X',A'), (X,A), and f.

5 LEMMA *Let (Y,B) be a pair and let Y' be a closed subset of Y. Assume that there is a homotopy H: $(Y,B) \times I \to (Y,B)$ such that*

(a) $H(y,0) = y$, *for* $y \in Y$.
(b) $H(Y' \times I) \subset Y'$.
(c) $H(Y' \times 1) = y_0$.

Then the collapsing map k: $(Y,B) \to (Y,B)/Y'$ is a homotopy equivalence.

PROOF Define a map f: $(Y,B)/Y' \to (Y,B)$ by the equation

$$f(k(y)) = H(y,1) \qquad y \in Y$$

[this is well-defined, because $H(Y' \times 1) = y_0$]. We show that f is a homotopy inverse of k. By definition of f, we see that H is a homotopy from $1_{(Y,B)}$ to $f \circ k$. On the other hand, because $H(Y' \times 1) \subset Y'$, there is a homotopy

$$H'\colon ((Y,B)/Y') \times I \to (Y,B)/Y'$$

such that $H'(k(y),t) = k(H(y,t))$ for $y \in Y$ and $t \in I$. Then

$$k(f(k(y))) = k(H(y,1)) = H'(k(y),1) \qquad y \in Y$$

Therefore H' is a homotopy from the identity map of $(Y,B)/Y'$ to $k \circ f$, and f is a homotopy inverse of k. ■

6 COROLLARY *Let f: $(X',A') \to (X,A)$ be a map and let i: $(X,A) \subset (C_{f'},C_{f''})$. Then $CX \subset C_{i'}$, $(C_{i'},C_{i''})/CX = (C_{f'},C_{f''})/X$, and the collapsing map*

$$k\colon (C_{i'},C_{i''}) \to (C_{i'},C_{i''})/CX$$

is a homotopy equivalence.

PROOF $C_{i'}$ is the quotient space of $CX' \vee CX$ with the identifications $[x',1] = [f(x'),1]$ for all $x' \in X'$, hence $CX \subset C_{i'}$. Since $C_{i'}$ is the union of the closed subspaces CX and $C_{f'}$, it follows that

$$C_{i'}/CX = C_{f'}/(C_{f'} \cap CX) = C_{f'}/X$$

Similarly, $C_{i''}/CA = C_{f''}/A$, and because $C_{i''} \cap CX = CA$,

$$(C_{i'},C_{i''})/CX = (C_{f'},C_{f''})/X$$

This proves the first two parts of the corollary.

Let F: $C(X,A) \times I \to C(X,A)$ be the contraction defined by $F([x,t], t') = [x, (1 - t')t]$ and let g: $C(X',A') \to (C_{i'},C_{i''})$ be the composite

$$C(X',A') \subset C(X',A') \vee C(X,A) \to (C_{i'},C_{i''})$$

where the second map is the canonical map. The composite

$$(X',A') \times I \xrightarrow{f \times 1} (X,A) \times I \subset C(X,A) \times I \xrightarrow{F} C(X,A) \subset (C_{i'},C_{i''})$$

is a homotopy $G\colon (X',A') \times I \to (C_{i'},C_{i''})$ such that $G(x',0) = [f(x'),1] = g[x',1]$. By lemma 2, there is a homotopy $F'\colon C(X',A') \times I \to (C_{i'}C_{i''})$ such that $F' \mid (X',A') \times I = G$ and $F'([x',t], 0) = g([x',t])$. Then a homotopy

$$H\colon (C_{i'},C_{i''}) \times I \to (C_{i'},C_{i''})$$

is defined by the equations

$$\begin{aligned} H([x',t], t') &= F'([x',t], t') \qquad & x' \in X';\ t, t' \in I \\ H([x,t], t') &= F([x,t], t') \qquad & x \in X;\ t, t' \in I \end{aligned}$$

[this is well-defined because $F'([x',1], t') = G(x',t') = F([f(x'),1], t')$]. Then H satisfies a, b, and c of lemma 5 with $(Y,B) = (C_{i'},C_{i''})$ and $Y' = CX$. Therefore $k\colon (C_{i'},C_{i''}) \to (C_{i'},C_{i''})/CX$ is a homotopy equivalence. ■

Recall from Sec. 1.6 that the suspension SX is defined as the space $X \times I/(X \times 0 \cup x_0 \times I \cup X \times 1)$ (therefore $SX = CX/X$). For a pair (X,A) we define $S(X,A) = (SX,SA)$. Then, for any map $f\colon (X',A') \to (X,A)$, we have $(C_{f'},C_{f''})/X = S(X',A')$, and we let $k\colon (C_{f'},C_{f''}) \to S(X',A')$ be the collapsing map.

7 LEMMA *For any map $f\colon (X',A') \to (X,A)$ the sequence*

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{k} S(X',A') \xrightarrow{Sf} S(X,A)$$

is coexact.

PROOF We shall use the coexact sequence 4,

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{j} (C_{i'},C_{i''}) \xrightarrow{l} (C_{j'},C_{j''})$$

By corollary 6, there is a homotopy equivalence

$$(C_{i'},C_{i''}) \xrightarrow{k'} (C_{f'},C_{f''})/X = S(X',A')$$

and the composite $(C_{f'},C_{f''}) \xrightarrow{j} (C_{i'},C_{i''}) \xrightarrow{k'} S(X',A')$ is seen to be the collapsing map $k\colon (C_{f'},C_{f''}) \to S(X',A')$. Also by corollary 6, there is a homotopy equivalence

$$(C_{j'},C_{j''}) \xrightarrow{k''} (C_{j'},C_{j''})/CC_{f'} = (C_{i'},C_{i''})/C_{f'} = S(X,A)$$

and the composite $(C_{i'},C_{i''}) \xrightarrow{l} (C_{j'},C_{j''}) \xrightarrow{k''} S(X,A)$ is easily seen to be the collapsing map $\bar{k}\colon (C_{i'},C_{i''}) \to (C_{i'},C_{i''})/C_{f'} = S(X,A)$. Let $g\colon S(X',A') \to S(X,A)$ be the map defined by $g([x',t]) = [f(x'), 1 - t]$. The triangle

$$\begin{tikzcd} & (C_{i'},C_{i''}) \arrow[dl, "k'"'] \arrow[dr, "\bar{k}"] & \\ S(X',A') \arrow[rr, "g"] & & S(X,A) \end{tikzcd}$$

is homotopy commutative because a homotopy

$$H\colon (C_{i'},C_{i''}) \times I \to S(X,A)$$

from $\bar{k}$ to $g \circ k'$ is defined by

$$\begin{aligned} H([x',t], t') &= [f(x'), 1 - tt'] \qquad & x' \in X';\ t, t' \in I \\ H([x,t], t') &= [x, (1 - t')t] \qquad & x \in X;\ t, t' \in I \end{aligned}$$

[this is well-defined because $H([x',1], t') = [f(x'), 1 - t'] = H([f(x'),1], t')]$. Therefore there is a homotopy-commutative diagram

$$\begin{tikzcd}
(C_{f'},C_{f''}) \arrow[r, "j"] \arrow[dr, "k"'] & (C_{i'},C_{i''}) \arrow[r, "l"] \arrow[d, "k'"] & (C_{j'},C_{j''}) \arrow[d, "k''"] \\
& S(X',A') \arrow[r, "g"] & S(X,A)
\end{tikzcd}$$

in which k' and k'' are homotopy equivalences. From the coexactness of the sequence 4, the coexactness of the sequence

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{k} S(X',A') \xrightarrow{g} S(X,A)$$

follows. Let h: $S(X,A) \to S(X,A)$ be the homeomorphism defined by $h([x,t]) = [x, 1 - t]$. The coexactness of the above sequence implies the coexactness of the sequence

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{k} S(X',A') \xrightarrow{h \circ g} S(X,A)$$

Because $h \circ g = Sf$, this is the desired result. ∎

8 LEMMA *If the sequence*

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{g} (X'',A'')$$

is coexact, so is the suspended sequence

$$S(X',A') \xrightarrow{Sf} S(X,A) \xrightarrow{Sg} S(X'',A'')$$

PROOF For any pair (Y,B) let $\Omega(Y,B) = (\Omega Y,\Omega B)$. By theorem 2.8 in the Introduction, there is a commutative diagram (in which the vertical maps are equivalences of pointed sets)

$$\begin{tikzcd}
{[S(X'',A''); Y,B]} \arrow[r, "(Sg)\#"] \arrow[d, leftrightarrow] & {[S(X,A); Y,B]} \arrow[r, "(Sf)\#"] \arrow[d, leftrightarrow] & {[S(X',A'); Y,B]} \arrow[d, leftrightarrow] \\
{[X'',A''; \Omega(Y,B)]} \arrow[r, "g\#"] & {[X,A; \Omega(Y,B)]} \arrow[r, "f\#"] & {[X',A'; \Omega(Y,B)]}
\end{tikzcd}$$

Hence im $(Sg)\# = \ker (Sf)\#$ in the top sequence is equivalent to im $g\# = \ker f\#$ in the bottom sequence. ∎

We define $S^n(X,A)$ inductively for $n \geq 0$ so that

$$S^0(X,A) = (X,A)$$
$$S^n(X,A) = S(S^{n-1}(X,A)) \qquad n \geq 1$$

9 THEOREM *For any map* f: $(X',A') \to (X,A)$ *the sequence*

$$(X',A') \xrightarrow{f} (X,A) \xrightarrow{i} \cdots \xrightarrow{S^nf} S^n(X,A) \xrightarrow{S^ni} S^n(C_{f'},C_{f''}) \xrightarrow{S^nk} S^{n+1}(X',A') \xrightarrow{S^{n+1}f} \cdots$$

is coexact.

PROOF From lemmas 7 and 8, for $n \geq 0$ there is a coexact sequence

$$S^n(X',A') \xrightarrow{S^nf} S^n(X,A) \xrightarrow{S^ni} S^n(C_{f'},C_{f''}) \xrightarrow{S^nk} S^{n+1}(X',A') \xrightarrow{S^{n+1}f} S^{n+1}(X,A)$$

Since every three-term subsequence of the sequence in the theorem is contained in one of these five-term coexact sequences, the result follows. ∎

In the coexact sequence of theorem 9 all but the first three pairs are H cogroup pairs, and all but the first three of these are abelian. Furthermore, all maps between H cogroup pairs are homomorphisms. Thus, for any (Y,B) the coexact sequence of homotopy classes of maps of the sequence of theorem 9 into the fixed pair (Y,B) (with B not necessarily closed in Y) consist of groups and homomorphisms except for the last three pointed sets, and all but three of the groups are abelian.

We now show how the last group in the sequence, namely $[S(X',A'); Y,B]$, acts as a group of operators on the left on the next set in the sequence, namely $[C_{f'},C_{f''}; Y,B]$, in such a way that the orbits are mapped injectively by $i\#$ into $[X,A; Y,B]$. If α: $S(X',A') \to (Y,B)$ and β: $(C_{f'},C_{f''}) \to (Y,B)$, we define

$$\alpha \top \beta\colon (C_{f'},C_{f''}) \to (Y,B)$$

by
$$(\alpha \top \beta)[x',t] = \begin{cases} \alpha[x',2t] & 0 \le t \le \tfrac{1}{2},\ x' \in X',\ t \in I \\ \beta[x', 2t-1] & \tfrac{1}{2} \le t \le 1,\ x' \in X',\ t \in I \end{cases}$$

and
$$(\alpha \top \beta)(x) = \beta(x) \qquad x \in X$$

It is then clear that $(\alpha \top \beta) \mid (X,A) = \beta \mid (X,A)$, and the following statements are easily verified.

10 *$\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ (or $\beta \simeq \beta'$ rel X) implies $\alpha \top \beta \simeq \alpha' \top \beta'$ (or $\alpha \top \beta \simeq \alpha' \top \beta'$ rel X).* ∎

11 *If α_0 is the constant map, then $\alpha_0 \top \beta \simeq \beta$ rel X.* ∎

12 $(\alpha_1 * \alpha_2) \top \beta \simeq \alpha_1 \top (\alpha_2 \top \beta)$ rel X. ∎

13 $\alpha_1 \top (\alpha_2 \circ k) \simeq (\alpha_1 * \alpha_2) \circ k$ rel X. ∎

Given maps β_1,β_2: $(C_{f'},C_{f''}) \to (Y,B)$ such that $\beta_1 \mid (X,A) = \beta_2 \mid (X,A)$, we define $d(\beta_1,\beta_2)$: $S(X',A') \to (Y,B)$ by

$$d(\beta_1,\beta_2)[x',t] = \begin{cases} \beta_1[x',2t] & 0 \le t \le \tfrac{1}{2},\ x' \in X',\ t \in I \\ \beta_2[x', 2-2t] & \tfrac{1}{2} \le t \le 1,\ x' \in X',\ t \in I \end{cases}$$

The following results are easily verified.

14 *$\beta_1 \simeq \beta_1'$ rel X and $\beta_2 \simeq \beta_2'$ rel X imply $d(\beta_1,\beta_2) \simeq d(\beta_1',\beta_2')$.* ∎

15 $d(\beta_1,\beta_3) \simeq d(\beta_1,\beta_2) * d(\beta_2,\beta_3)$. ∎

16 $d(\alpha \top \beta,\beta) \simeq \alpha$. ∎

17 $\beta_1 \simeq d(\beta_1,\beta_2) \top \beta_2$ rel X. ∎

From statements 17, 10, and 11, it follows that if $d(\beta_1,\beta_2)$ is null homotopic, then $\beta_1 \simeq \beta_2$ rel X. Conversely, if $\beta_1 \simeq \beta_2$ rel X, it follows from statements 11, 14, and 16 that

$$d(\beta_1,\beta_2) \simeq d(\alpha_0 \top \beta_1,\beta_1) \simeq \alpha_0$$

Therefore we have $\beta_1 \simeq \beta_2$ rel X if and only if $d(\beta_1,\beta_2)$ is null homotopic.

By statements 10, 11, and 12, there is an action of $[S(X',A'); Y,B]$ on the left on $[C_{f'},C_{f''}; Y,B]$ defined by $[\alpha] \top [\beta] = [\alpha \top \beta]$.

18 THEOREM *Given $[\beta_1]$, $[\beta_2] \in [C_{f'},C_{f''}; Y,B]$, then $i\#[\beta_1] = i\#[\beta_2]$ if and only if there is $[\alpha] \in [S(X',A'); Y,B]$ such that $[\beta_1] = [\alpha] \top [\beta_2]$.*

PROOF By the definition of $\alpha \top \beta_2$ we see that

$$i\#[\alpha \top \beta_2] = [(\alpha \top \beta_2) \mid (X,A)] = [\beta_2 \mid (X,A)] = i\#[\beta_2]$$

showing that $i\#([\alpha] \top [\beta_2]) = i\#[\beta_2]$. Conversely, if $i\#[\beta_1] = i\#[\beta_2]$, we can choose representatives β_1 and β_2 such that $\beta_1 \mid (X,A) = \beta_2 \mid (X,A)$ [because the map i: $(X,A) \subset (C_{f'},C_{f''})$ is a cofibration]. Then, by statement 17,

$$[\beta_1] = [d(\beta_1,\beta_2) \top \beta_2] = [d(\beta_1,\beta_2)] \top [\beta_2] \quad \blacksquare$$

19 THEOREM *Given $[\alpha_1]$, $[\alpha_2] \in [S(X',A'); Y,B]$, then $k\#[\alpha_1] = k\#[\alpha_2]$ if and only if there is $[\gamma] \in [S(X,A); Y,B]$ such that $[\alpha_2] = [\alpha_1] + (Sf)\#[\gamma]$.*

PROOF By statement 13, if β_0: $(C_{f'},C_{f''}) \to (Y,B)$ is the constant map

$$\begin{aligned} k\#[\alpha_1 * (\gamma \circ Sf)] &= [\alpha_1] \top (k\#Sf\#[\gamma]) = [\alpha_1] \top [\beta_0] \\ &= [\alpha_1] \top k\#[\alpha_0] = k\#[\alpha_1 * \alpha_0] \end{aligned}$$

Therefore $k\#([\alpha_1] + (Sf)\#[\gamma]) = k\#[\alpha_1]$. Conversely, if $k\#[\alpha_1] = k\#[\alpha_2]$, then by statements 10 and 13,

$$0 = k\#[\alpha_1{}^{-1} * \alpha_1] = [\alpha_1{}^{-1}] \top k\#[\alpha_1] = [\alpha_1{}^{-1}] \top k\#[\alpha_2] = k\#[\alpha_1{}^{-1} * \alpha_2]$$

Therefore there is $[\gamma] \in [S(X,A); Y,B]$ such that $[\alpha_1{}^{-1} * \alpha_2] = (Sf)\#[\gamma]$, and so

$$[\alpha_2] = [\alpha_1] + [\alpha_1{}^{-1} * \alpha_2] = [\alpha_1] + (Sf)\#[\gamma] \quad \blacksquare$$

2 HIGHER HOMOTOPY GROUPS

The higher homotopy groups of a space or pair are covariant functors defined to be sets of homotopy classes of maps of fixed spaces or pairs into the given one. In the absolute case these are the functors already defined in Sec. 1.6. The exactness property established in the last section implies an important exactness property relating relative and absolute homotopy groups. This section contains definitions of the homotopy groups, some of their elementary properties, and a proof of the exactness of the homotopy sequence of a fibration.

We shall use 0 as base point for I and for the subspace $\dot I \subset I$. Let X be a space with base point x_0. For $n \geq 1$ the homotopy group $\pi_n(X)$ [or $\pi_n(X,x_0)$, when it is important to indicate the base point] is the group $[S^n(\dot I);X]$ [this being equivalent to the definition given in Sec. 1.6, because S^n is homeomorphic to $S^n(S^0) \approx S^n(\dot I)$]. For $n = 0$ the homotopy set $\pi_0(X)$ is defined to be the pointed set $[\dot I;X]$ (that is, the set of path components of X with the path component of x_0 as distinguished element). Then π_n is a covariant functor from the category of pointed spaces to the category of abelian groups if $n \geq 2$, the category of groups if $n = 1$, and the category of pointed sets if $n = 0$.

Let (X,A) be a pair with base point $x_0 \in A$. For $n \geq 1$ the *nth relative homotopy group* (or *homotopy set* for $n = 1$), denoted by $\pi_n(X,A)$ or $\pi_n(X,A,x_0)$, is defined to equal $[S^{n-1}(I,\dot{I}); X,A]$. Then π_n is a covariant functor from the category of pairs of pointed spaces to the category of abelian groups if $n \geq 3$, the category of groups if $n = 2$, and the category of pointed sets if $n = 1$.

There is a homeomorphism of $S(\dot{I})$ with $I/\dot{I}$ which sends $[0,t] \in S(\dot{I})$ to the base point of $I/\dot{I}$ and $[1,t] \in S(\dot{I})$ to that point of $I/\dot{I}$ determined by the point $t \in I$. Therefore, for $n \geq 1$, $S^n(\dot{I})$ and $S^{n-1}(I/\dot{I}) = S^{n-1}(I)/S^{n-1}(\dot{I})$ are homeomorphic. This induces a natural one-to-one correspondence between $[S^{n-1}(I,\dot{I}); X,\{x_0\}]$ and $[S^n(\dot{I});X]$. By means of this correspondence we identify the relative homotopy group $\pi_n(X,\{x_0\})$ for $n \geq 1$ with the absolute homotopy group $\pi_n(X)$. Then the inclusion map j: $(X,\{x_0\}) \subset (X,A)$ induces a homomorphism

$$j_\#: \pi_n(X) \to \pi_n(X,A) \qquad n \geq 1$$

Because $S^n(\dot{I})$ is path connected for $n \geq 1$, it follows that if X' is the path component of X containing x_0, the inclusion map $X' \subset X$ induces isomorphisms $\pi_n(X') \approx \pi_n(X)$ for $n \geq 1$. Similarly, if A' is the path component of A containing x_0, the inclusion map $(X',A') \subset (X,A)$ induces isomorphisms $\pi_n(X',A') \approx \pi_n(X,A)$ for $n \geq 1$.

We present an alternative description of the relative homotopy groups. For $n \geq 1$ there is a homeomorphism of $S^{n-1}(I,\dot{I})$ with $(I \times I^{n-1}, \dot{I} \times I^{n-1})/(I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$ sending $[\cdots[t,t_1], \ldots, t_{n-1}]$ to $[t,t_1, \ldots, t_{n-1}]$ (I^0 is a single point and $\dot{I}^0$ is empty). Therefore, for $n \geq 1$, $\pi_n(X,A,x_0)$ is in one-to-one correspondence with the set of homotopy classes of maps

$$(I^n, \dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1}) \to (X,A,x_0)$$

Since $I \times \dot{I}^{n-1} \cup 0 \times I^{n-1}$ is contractible, if $z_0 = (0,0, \ldots, 0)$, the inclusion map

$$(I^n,\dot{I}^n,z_0) \subset (I^n, \dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$$

is a homotopy equivalence. Hence, for $n \geq 1$, $\pi_n(X,A,x_0)$ is in one-to-one correspondence with the set of homotopy classes of maps

$$(I^n,\dot{I}^n,z_0) \to (X,A,x_0)$$

Since $(I^n,\dot{I}^n,z_0)$ is homeomorphic to (E^n,S^{n-1},p_0) for $n \geq 1$, $\pi_n(X,A,x_0)$ is in one-to-one correspondence with the set of homotopy classes of maps

$$(E^n,S^{n-1},p_0) \to (X,A,x_0)$$

The following condition for a map $(E^n,S^{n-1},p_0) \to (X,A,x_0)$ to represent the trivial element of $\pi_n(X,A,x_0)$ is a relative version of theorem 1.6.7.

1 THEOREM *Given a map* α: $(E^n,S^{n-1},p_0) \to (X,A,x_0)$, *then* $[\alpha] = 0$ *in*

$\pi_n(X,A,x_0)$ if and only if α is homotopic relative to S^{n-1} to some map of E^n to A.

PROOF Assume $[\alpha] = 0$ in $\pi_n(X,A,x_0)$. Then there is a homotopy

$$H: (E^n,S^{n-1},p_0) \times I \to (X,A,x_0)$$

from α to the constant map $E^n \to x_0$. A homotopy H' relative to S^{n-1} from α to some map E^n to A is defined by

$$H'(z,t) = \begin{cases} H\left(\dfrac{z}{1 - t/2}, t\right) & 0 \le \|z\| \le 1 - \dfrac{t}{2} \\ H\left(\dfrac{z}{\|z\|}, 2 - 2\|z\|\right) & 1 - \dfrac{t}{2} \le \|z\| \le 1 \end{cases}$$

Conversely, if α is homotopic relative to S^{n-1} to some map α' such that $\alpha'(E^n) \subset A$, then $[\alpha] = [\alpha']$ in $\pi_n(X,A,x_0)$, and it suffices to show that $[\alpha'] = 0$ in $\pi_n(X,A,x_0)$. A homotopy $H: (E^n,S^{n-1},p_0) \times I \to (X,A,x_0)$ from α' to the constant map $E^n \to x_0$ is defined by

$$H(z,t) = \alpha'((1 - t)z + tp_0) \quad \blacksquare$$

A pair (X,A) is said to be *n-connected* for $n \ge 0$ if for $0 \le k \le n$ every map $\alpha: (E^k,S^{k-1}) \to (X,A)$ is homotopic relative to S^{k-1} to some map of E^k to A. For $k = 0$, (E^0,S^{-1}) is a pair consisting of a single point and the empty set, and the condition that every map $\alpha: (E^0,S^{-1}) \to (X,A)$ be homotopic to a map $E^0 \to A$ is equivalent to the condition that every point of X be joined by a path to some point of A. From theorem 1 we obtain the following relation between n-connectedness of (X,A) and the vanishing of relative homotopy groups of (X,A).

2 COROLLARY *A pair (X,A) is n-connected for $n \ge 0$ if and only if every path component of X intersects A and for every point $a \in A$ and every $1 \le k \le n$, $\pi_k(X,A,a) = 0$.* $\blacksquare$

For $n \ge 1$ there is a map (which is a homomorphism for $n \ge 2$)

$$\partial: \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0)$$

defined by restriction. That is, given $\alpha: S^{n-1}(I,\dot{I}) \to (X,A)$, then

$$\partial[\alpha] = [\alpha \mid S^{n-1}(\dot{I})]$$

It is trivial that if $f: (X',A',x_0') \to (X,A,x_0)$ is a map, there is a commutative square

$$\begin{array}{ccc} \pi_n(X',A',x_0') & \xrightarrow{\partial} & \pi_{n-1}(A',x_0') \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle (f|A')_\#} \\ \pi_n(X,A,x_0) & \xrightarrow{\partial} & \pi_n(A,x_0) \end{array}$$

In other words, ∂ is a natural transformation between covariant functors $\pi_n(X,A)$ and $\pi_{n-1}(A)$ on the category of pairs (X,A) of pointed spaces. Thus the homotopy-group functors and the natural transformation ∂ are in analogy with the constituents of a homology theory. We shall show that they also satisfy many of the axioms of homology theory. It is obvious that the homotopy axiom and the dimension axiom are satisfied for the homotopy-group functors.

We shall now investigate the exactness property. Given a pair (X,A) of pointed spaces, let $i: A \subset X$ and $j: (X,\{x_0\}) \subset (X,A)$. The *homotopy sequence of* (X,A) [or of (X,A,x_0)] is the sequence of pointed sets (all but the last three being groups)

$$\cdots \to \pi_{n+1}(X,A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_\#} \pi_n(X) \xrightarrow{j_\#} \pi_n(X,A) \xrightarrow{\partial} \cdots \xrightarrow{i_\#} \pi_0(X)$$

3 THEOREM *The homotopy sequence of a pair is exact.*

PROOF Let $f: (\dot{I},\{0\}) \subset (\dot{I},\dot{I})$ and let $f': \dot{I} \subset \dot{I}$ and $f'': \{0\} \subset \dot{I}$. By theorem 7.1.9, there is a coexact sequence

$$(\dot{I},\{0\}) \xrightarrow{f} (\dot{I},\dot{I}) \xrightarrow{i} (C_{f'},C_{f''}) \xrightarrow{k} S(\dot{I},\{0\}) \xrightarrow{Sf} S(\dot{I},\dot{I}) \to \cdots$$

Let $g: (C_{f'},C_{f''}) \to (I,\dot{I})$ be the homeomorphism defined by $g([0,t]) = 0$ and $g([1,t]) = t$. Then the composite $g \circ i$ is the inclusion map $i': (\dot{I},\dot{I}) \subset (I,\dot{I})$, and the composite $k \circ g^{-1}$ equals the composite

$$(I,\dot{I}) \xrightarrow{k'} (I/\dot{I},\{0\}) \xrightarrow{h} (S(\dot{I}),\{0\})$$

where k' is the collapsing map and h is the homeomorphism used in identifying $\pi_n(X,\{x_0\})$ with $\pi_n(X)$. Therefore there is a coexact sequence

$$(\dot{I},\{0\}) \xrightarrow{f} (\dot{I},\dot{I}) \xrightarrow{i'} (I,\dot{I}) \xrightarrow{h \circ k'} S(\dot{I},\{0\}) \xrightarrow{Sf} \cdots$$

This yields an exact sequence

$$\cdots \to \pi_{n+1}(X,A) \xrightarrow{(S^n i')^\#} \pi_n(A) \xrightarrow{(S^n f)^\#} \pi_n(X) \xrightarrow{(S^{n-1}(h \circ k'))^\#} \pi_n(X,A) \to \cdots \to \pi_0(X)$$

The proof is completed by the trivial verification that

$$(S^n i')^\# = \partial,\ (S^n f)^\# = i_\#, \text{ and } \quad (S^{n-1}(h \circ k'))^\# = j_\# \quad \blacksquare$$

4 COROLLARY *For $n \geq 0$, (E^{n+1},S^n) is n-connected.*

PROOF For $n \geq 0$, E^{n+1} is path connected and S^n is nonempty; therefore every path component of E^{n+1} meets S^n. If $x \in S^n$, then $\pi_k(E^{n+1},x) = 0$ for $0 \leq k$, because E^{n+1} is contractible. By theorem 3.4.11, $\pi_k(S^n,x) = 0$ if $0 \leq k < n$. It follows from theorem 3 that $\pi_k(E^{n+1},S^n,x) = 0$ for $1 \leq k \leq n$. The result follows from corollary 2. $\blacksquare$

We shall see that the excision property fails to hold for the homotopy group functors. There is, however, a different property possessed by the homotopy group functors but not by homology functors. This property is the existence of an isomorphism between the absolute homotopy groups of the base space of a fibration and the corresponding relative homotopy groups of the total space modulo the fiber. This is true for a more general class of maps than fibrations, and we now present the relevant definition.

A map $p: E \to B$ is called a *weak fibration* (or *Serre fiber space* in the literature) if p has the homotopy lifting property with respect to the collection of cubes $\{I^n\}_{n\geq 0}$. E is called the *total space* and B the *base space* of the weak fibration. For $b \in B$, $p^{-1}(b)$ is called the *fiber over* b.

If s is a simplex, $|s|$ is homeomorphic to some cube, and so a map $p: E \to B$ is a weak fibration if and only if it has the homotopy lifting property with respect to the space of any simplex. We shall show that, in fact, a weak fibration has the homotopy lifting property with respect to any polyhedron.

It is clear that a fibration is a weak fibration. If $p: E \to B$ is a weak fibration and $f: B' \to B$ is a map, let E' be the fibered product of B' and E. Then there is a weak fibration $p': E' \to B'$, called the *weak fibration induced from p by f.*

5 LEMMA *Let $p: E \to B$ be a weak fibration and let $g: I^n \times 0 \cup \dot{I}^n \times I \to E$ and $H: I^n \times I \to B$ be maps, with $n \geq 0$, such that H is an extension of $p \circ g$. Then there is a map $G: I^n \times I \to E$ such that $p \circ G = H$ and G is an extension of g.*

PROOF The lemma asserts that the dotted arrow in the diagram

$$\begin{tikzcd} I^n \times 0 \cup \dot{I}^n \times I \arrow[r, "g"] \arrow[d, "\cap"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"] \arrow[ur, dashed] & B \end{tikzcd}$$

represents a map making the diagram commutative. This follows from the homotopy lifting property of p since the pair $(I^n \times I, I^n \times 0 \cup \dot{I}^n \times I)$ is homeomorphic to the pair $(I^n \times I, I^n \times 0)$. ∎

6 THEOREM *Let (X,A) be a polyhedral pair and let $p: E \to B$ be a weak fibration. Given maps $g: X \times 0 \cup A \times I \to E$ and $H: X \times I \to B$ such that H is an extension of $p \circ g$, there is a map $G: X \times I \to E$ such that $p \circ G = H$ and G is an extension of g.*

PROOF The method of obtaining G involves a standard stepwise-extension procedure over the successive skeleta of a triangulation of X. Let (K,L) be a triangulation of (X,A) and identify (X,A) with $(|K|,|L|)$. For $q \geq -1$ set $X_q = |K| \times 0 \cup (|K^q \cup L| \times I)$, so that $X_{-1} = X \times 0 \cup A \times I$ and $X_{q-1} \subset X_q$ for $q \geq 0$. By induction on q, we shall define a sequence of maps $G_q: X_q \to E$ such that

(a) $G_{-1} = g$
(b) $G_q \mid X_{q-1} = G_{q-1}$ for $q \geq 0$
(c) $p \circ G_q = H \mid X_q$ for $q \geq -1$

Once a sequence $\{G_q\}$ with these properties is obtained, a map $G: X \times I \to E$ with the desired properties is defined by the conditions $G \mid X_q = G_q$, for $q \geq -1$. Thus the problem is reduced to the construction of such a sequence $\{G_q\}$.

Condition (*a*) defines G_{-1}. Assume G_q defined for $q < n$, where $n \geq 0$. To define G_n to satisfy conditions (*b*) and (*c*), for every n-simplex $s \in K - L$ let $g_s\colon |s| \times 0 \cup |\dot{s}| \times I \to E$ and $H_s\colon |s| \times I \to B$ be the maps defined by $g_s = G_{n-1} \mid (|s| \times 0 \cup |\dot{s}| \times I)$ and $H_s = H \mid (|s| \times I)$. Because $(|s|,|\dot{s}|)$ is homeomorphic to $(I^n,\dot{I}^n)$, it follows from lemma 5 that there is a map $G_s\colon |s| \times I \to E$ such that $G_s \mid (|s| \times 0 \cup |\dot{s}| \times I) = g_s$ and $p \circ G_s = H_s$. Then a map $G_n\colon X_n \to E$ satisfying conditions (*b*) and (*c*) is defined by the conditions $G_n \mid X_{n-1} = G_{n-1}$ and $G_n \mid (|s| \times I) = G_s$ for s an n-simplex of $K - L$. ∎

Taking A to be empty in theorem 6, we see that a weak fibration has the homotopy lifting property with respect to any polyhedron.

7 COROLLARY *Let (X',A') be a polyhedral pair such that A' is a strong deformation retract of X' and let $p\colon E \to B$ be a weak fibration. Given maps $g'\colon A' \to E$ and $H'\colon X' \to B$ such that $H' \mid A' = p \circ g'$, there is a map $G'\colon X' \to E$ such that $p \circ G' = H'$ and $G' \mid A' = g'$.*

PROOF Let $D\colon X' \times I \to X'$ be a strong deformation retraction of X' to A'. Then $D(X' \times 1 \cup A' \times I) \subset A'$, and we define $g\colon X' \times 1 \cup A' \times I \to E$ to be the composite

$$X' \times 1 \cup A' \times I \xrightarrow{D} A' \xrightarrow{g'} E'$$

Let $H\colon X' \times I \to B$ be the composite

$$X' \times I \xrightarrow{D} X' \xrightarrow{H'} B$$

Then H is an extension of $p \circ g$, and it follows from theorem 6 that there is a map $G\colon X' \times I \to E$ such that $p \circ G = H$ and G is an extension of g. Let $G'\colon X' \to E$ be defined by $G'(x') = G(x',0)$. Then G' has the desired properties. ∎

The following theorem is the main result relating the homotopy groups of the base and total space of a weak fibration.

8 THEOREM *Let $p\colon E \to B$ be a weak fibration and suppose $b_0 \in B' \subset B$. Let $E' = p^{-1}(B')$ and let $e_0 \in p^{-1}(b_0)$. Then p induces a bijection*

$$p_\#\colon \pi_n(E,E',e_0) \approx \pi_n(B,B',b_0) \qquad n \geq 1$$

PROOF To show that $p_\#$ is surjective, let $\alpha\colon (I^n,\dot{I}^n,z_0) \to (B,B',b_0)$ represent an element of $\pi_n(B,B',b_0)$. Because z_0 is a strong deformation retract of I^n, we can apply corollary 7 to the pair $(I^n,\{z_0\})$ and to maps $g'\colon \{z_0\} \to E$ and $H'\colon I^n \to B$, where $g'(z_0) = e_0$ and $H' = \alpha \mid I^n$. We then obtain a map $G'\colon I^n \to E$ such that $p \circ G' = H'$ and $G'(z_0) = e_0$. Then

$$G'(\dot{I}^n) \subset p^{-1}(H'(\dot{I}^n)) \subset p^{-1}(B') = E'$$

Therefore G' defines a map $\alpha'\colon (I^n,\dot{I}^n,z_0) \to (E,E',e_0)$ such that $p \circ \alpha' = \alpha$. Then α' represents an element $[\alpha'] \in \pi_n(E,E',e_0)$ and $p_\#[\alpha'] = [\alpha]$.

To show that $p_\#$ is injective, let $\alpha_0,\alpha_1\colon (I^n,\dot{I}^n,z_0) \to (E,E',e_0)$ be such that $p \circ \alpha_0 \simeq p \circ \alpha_1$. Let $X' = I^n \times I$ and $A' = (I^n \times 0) \cup (z_0 \times I) \cup (I^n \times 1)$.

Then (X',A') is a polyhedral pair, and because X' and A' are both contractible, A' is a strong deformation retract of X'. Let $g'\colon A' \to E$ be defined by $g'(z,0) = \alpha_0(z)$, $g'(z,1) = \alpha_1(z)$, and $g'(z_0,t) = e_0$ and let $H'\colon X' \to B$ be a map which is a homotopy from $p \circ \alpha_0$ to $p \circ \alpha_1$ in (B,B',b_0). By corollary 7, there is a map $G'\colon X' \to E$ such that $p \circ G' = H'$ and $G' \mid A' = g'$. Since

$$G'(\dot{I}^n \times I) \subset p^{-1}(H'(\dot{I}^n \times I)) \subset p^{-1}(B') = E'$$

G' is a homotopy from α_0 to α_1 in (E,E',e_0); hence $[\alpha_0] = [\alpha_1]$ in $\pi_n(E,E',e_0)$. ∎

9 COROLLARY *Let $p\colon E \to B$ be a weak fibration, $b_0 \in B$, and $e_0 \in F = p^{-1}(b_0)$. Then p induces a bijection*

$$p_\#\colon \pi_n(E,F,e_0) \approx \pi_n(B,b_0) \qquad n \geq 1$$

PROOF This follows from theorem 8 on taking $B' = \{b_0\}$ and using the canonical identification $\pi_n(B,\{b_0\},b_0) = \pi_n(B,b_0)$. ∎

If $p\colon E \to B$ is a weak fibration with $F = p^{-1}(b_0)$ and $e_0 \in F$, we define

$$\bar{\partial}\colon \pi_n(B,b_0) \to \pi_{n-1}(F,e_0) \qquad n \geq 1$$

to be the composite

$$\pi_n(B,b_0) \xrightarrow{p_\#^{-1}} \pi_n(E,F,e_0) \xrightarrow{\partial} \pi_{n-1}(F,e_0)$$

The *homotopy sequence of the weak fibration* is the sequence

$$\cdots \to \pi_n(F,e_0) \xrightarrow{i_\#} \pi_n(E,e_0) \xrightarrow{p_\#} \pi_n(B,b_0) \xrightarrow{\bar{\partial}} \pi_{n-1}(F,e_0) \to \cdots \xrightarrow{p_\#} \pi_0(B,b_0)$$

where $i\colon (F,e_0) \subset (E,e_0)$.

10 THEOREM *The homotopy sequence of a weak fibration is exact.*

PROOF Exactness at $\pi_0(E,e_0)$ is an easy consequence of the homotopy lifting property. Exactness at any set to the left of $\pi_0(E,e_0)$ is a consequence of the exactness of the homotopy sequence of the pair (E,F). ∎

11 COROLLARY *Let $p\colon E \to B$ be a weak fibration with unique path lifting. Then p induces an isomorphism*

$$p_\#\colon \pi_q(E,e_0) \approx \pi_q(B,p(e_0)) \qquad q \geq 2$$

PROOF Because $F = p^{-1}(p(e_0))$ has no nonconstant paths (by theorem 2.2.5), $\pi_q(F,e_0) = 0$ for $q \geq 1$. The result then follows from theorem 10. ∎

12 COROLLARY *For $q \geq 2$, $\pi_q(S^1) = 0$.*

PROOF This follows from application of corollary 11 to the covering projection $p\colon \mathbf{R} \to S^1$ and the fact that because $\mathbf{R}$ is contractible, $\pi_q(\mathbf{R}) = 0$ for all $q \geq 0$. ∎

13 COROLLARY *Let $p\colon S^{2n+1} \to P_n(\mathbf{C})$ be the Hopf fibration. Then p induces an isomorphism*

$$p_\#\colon \pi_q(S^{2n+1}) \approx \pi_q(P_n(\mathbf{C})) \qquad q \geq 3$$

PROOF Because $F = S^1$ for the Hopf fibration, this follows from corollary 12 and theorem 10. ■

14 COROLLARY $\pi_3(S^2) \neq 0$.

PROOF Because the identity map $(S^3,p_0) \subset (S^3,p_0)$ induces a nontrivial homomorphism of $H_3(S^3,p_0)$, it is not homotopic to the constant map. Therefore $\pi_3(S^3) \neq 0$, and the result follows from corollary 13, with $n = 1$ (since $P_1(\mathbf{C}) \approx S^2$). ■

This last result shows that, unlike the homology groups, the homotopy groups of a polyhedron need not vanish in degrees larger than the dimension of the polyhedron.

If H is a closed hemisphere of S^2 and a is the pole in H, then the pair $(S^2 - a, H - a)$ has the same homotopy type as (E^2,S^1). Therefore

$$\pi_3(S^2 - a, H - a) \approx \pi_3(E^2,S^1) \overset{\partial}{\approx} \pi_2(S^1) = 0$$

On the other hand, (S^2,H) has the same homotopy type as $(S^2,\{a\})$. Therefore

$$\pi_3(S^2,H) \approx \pi_3(S^2,\{a\}) = \pi_3(S^2) \neq 0$$

Hence we see that the excision map j: $(S^2 - a, H - a) \subset (S^2,H)$ does not induce an isomorphism of $\pi_3(S^2 - a, H - a)$ with $\pi_3(S^2,H)$. Therefore the excision property does not hold for homotopy groups.

Recall the path fibration p: $P(X,x_0) \to X$ with fiber $p^{-1}(x_0) = \Omega X$ (see corollary 2.8.8). Since $P(X,x_0)$ is contractible (by lemma 2.4.3), $\pi_n(P(X,x_0)) = 0$ for $n \geq 0$, and by theorem 10, there is an isomorphism

$$\bar{\partial}: \pi_n(X) \approx \pi_{n-1}(\Omega X) \qquad n \geq 1$$

This result can also be deduced directly from the canonical one-to-one correspondence $[S^n(\dot{I});X] \approx [S^{n-1}(\dot{I});\Omega X]$ given by the exponential law. We shall use the path space to prove the exactness of the homotopy sequence of a triple.

Given a triple (X,A,B) with base point $x_0 \in B$, let i: $(A,B) \subset (X,B)$ and j: $(X,B) \subset (X,A)$ and let j': $(A,\{x_0\}) \subset (A,B)$. Define

$$\partial': \pi_n(X,A,x_0) \to \pi_{n-1}(A,B,x_0) \qquad n \geq 2$$

to equal the composite

$$\pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \xrightarrow{j'_\#} \pi_{n-1}(A,B,x_0)$$

The *homotopy sequence of the triple* (X,A,B) is defined to be the sequence

$$\cdots \to \pi_{n+1}(X,A) \xrightarrow{\partial'} \pi_n(A,B) \xrightarrow{i_\#} \pi_n(X,B) \xrightarrow{j_\#} \pi_n(X,A) \to \cdots \to \pi_1(X,A)$$

15 THEOREM *The homotopy sequence of a triple is exact.*

PROOF Let p: $P(X,x_0) \to X$ be the path fibration and let $X' = P(X,x_0)$, $A' = p^{-1}(A)$, and $B' = p^{-1}(B)$. Then (X',A',B') is a triple, and it follows from theorem 8 that $p_\#$ maps the homotopy sequence of (X',A',B') bijectively to the homotopy sequence of (X,A,B). Therefore it suffices to prove that the homotopy sequence of the triple (X',A',B') is exact.

Let i: $(A',B') \subset (X',B')$, j: $(X',B') \subset (X',A')$, i': $B' \subset A'$, and j': $A' \subset (A',B')$. There is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & \pi_{n+1}(X',A') & \xrightarrow{\partial'} & \pi_n(A',B') & \xrightarrow{i_\#} & \pi_n(X',B') & \xrightarrow{j_\#} & \pi_n(X',A') & \to \cdots \\
& \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & \\
\cdots \to & \pi_n(A') & \xrightarrow{j'_\#} & \pi_n(A',B') & \xrightarrow{\partial} & \pi_{n-1}(B') & \xrightarrow{i_\#} & \pi_{n-1}(A') & \to \cdots
\end{array}$$

in which each vertical map is a bijection (because X' is contractible). Therefore the exactness of the homotopy sequence of the triple (X',A',B') follows from the exactness of the homotopy sequence of the pair (A',B'). ■

This result can also be derived from the exactness of the homotopy sequence of a pair and functorial properties of the homotopy groups (as was the case with the corresponding exactness property for homology, theorem 4.8.5).

3 CHANGE OF BASE POINTS

The absolute and relative homotopy groups are defined for pointed spaces and pairs. This section is devoted to a study of the extent to which these groups depend on the choice of base point. By generalizing the methods of Sec. 1.8, we shall see that these groups based at different base points in the same path component are isomorphic, but the isomorphism between them is not usually unique. Much of these considerations apply to more general homotopy sets, and we begin with this.

Let (X,A) be a pair with base point $x_0 \in A$ and let (Y,B) be a pair. Two maps α_0, α_1: $(X,A) \to (Y,B)$ are said to be *freely homotopic* if they are homotopic as maps of (X,A) to (Y,B) (that is, no restriction is placed on the base point during the homotopy). If ω is a path in B from $\alpha_0(x_0)$ to $\alpha_1(x_0)$, an *ω-homotopy from α_0 to α_1* is a homotopy

$$H: (X,A) \times I \to (Y,B)$$

such that $H(x,0) = \alpha_0(x)$, $H(x,1) = \alpha_1(x)$, and $H(x_0,t) = \omega(t)$. If such a homotopy exists, we say that α_0 is *ω-homotopic* to α_1. It is clear that α_0 and α_1 are freely homotopic if and only if there is some path ω in B such that α_0 and α_1 are ω-homotopic. In particular, two maps α_0, α_1: $(X,A,x_0) \to (Y,B,y_0)$ are freely homotopic if and only if there is some closed path ω in B at y_0 such that α_0 is ω-homotopic to α_1.

Although the relation of free homotopy is an equivalence relation in the set of maps from (X,A) to (Y,B), for a fixed ω the relation of ω-homotopy is not generally an equivalence relation. For example, if ω is not a closed path, it is impossible for any map α_0 to be ω-homotopic to itself.

1 LEMMA (*a*) *Given a map f: $(X',A',x_0') \to (X,A,x_0)$, maps α_0, α_1: $(X,A) \to (Y,B)$, and a path ω in B such that α_0 is ω-homotopic to α_1, then $\alpha_0 \circ f$ is ω-homotopic to $\alpha_1 \circ f$.*

(*b*) *Given a map* $g: (Y,B) \to (Y',B')$, *maps* $\alpha_0, \alpha_1: (X,A) \to (Y,B)$, *and a path* ω *in* B *such that* α_0 *is* ω*-homotopic to* α_1, *then* $g \circ \alpha_0$ *is* $(g \circ \omega)$*-homotopic to* $g \circ \alpha_1$.

(*c*) *Given maps* $\alpha_0, \alpha'_0: (SX,SA,x_0) \to (Y,B,\omega(0))$ *and* $\alpha_1, \alpha'_1: (SX,SA,x_0) \to (Y,B,\omega(1))$ *such that* α_0 *is* ω*-homotopic to* α_1 *and* α'_0 *is* ω*-homotopic to* α'_1, *then* $\alpha_0 * \alpha'_0$ *is* ω*-homotopic to* $\alpha_1 * \alpha'_1$.

PROOF If $H: (X,A) \times I \to (Y,B)$ is an ω-homotopy from α_0 to α_1, then for (*a*) the composite

$$(X',A') \times I \xrightarrow{f \times 1} (X,A) \times I \xrightarrow{H} (Y,B)$$

is an ω-homotopy from $\alpha_0 \circ f$ to $\alpha_1 \circ f$, and for (*b*) the composite

$$(X,A) \times I \xrightarrow{H} (Y,B) \xrightarrow{g} (Y',B')$$

is a $(g \circ \omega)$-homotopy from $g \circ \alpha_0$ to $g \circ \alpha_1$.

In (*c*), if $H, H': (SX,SA) \times I \to (Y,B)$ are ω-homotopies from α_0 and α'_0 to α_1 and α'_1, respectively, the map

$$H * H': (SX,SA) \times I \to (Y,B)$$

defined by

$$(H * H')([x,t], t') = \begin{cases} H([x,2t], t') & 0 \le t \le \frac{1}{2} \\ H'([x, 2t-1], t') & \frac{1}{2} \le t \le 1 \end{cases}$$

is an ω-homotopy from $\alpha_0 * \alpha'_0$ to $\alpha_1 * \alpha'_1$. ∎

The base point x_0 for a pair (X,A) is said to be a *nondegenerate base point* if the inclusion map $(x_0,x_0) \subset (X,A)$ is a cofibration [that is, if, given a map $\alpha_0: (X,A) \to (Y,B)$ and a homotopy $\omega: x_0 \times I \to B$, there is a homotopy $H: (X,A) \times I \to (Y,B)$ such that $H(x,0) = \alpha_0(x)$ and $H(x_0,t) = \omega(t)$]. It follows from lemma 3.8.1 and corollary 3.2.4 that any point of a polyhedral pair is a nondegenerate base point.

2 LEMMA *Let* (X,A) *be a pair with nondegenerate base point and let* (Y,B) *be an arbitrary pair.*

(*a*) *Given a path* ω *in* B *and a map* $\alpha_1: (X,A,x_0) \to (Y,B,\omega(1))$, *there is a map* $\alpha_0: (X,A,x_0) \to (Y,B,\omega(0))$ *such that* α_0 *is* ω*-homotopic to* α_1.

(*b*) *If* $\alpha_0, \alpha'_0: (X,A,x_0) \to (Y,B,\omega(0))$ *are both* ω*-homotopic to* α_1, *then* $[\alpha_0] = [\alpha'_0]$ *in* $[X,A,x_0; Y,B,\omega(0)]$.

(*c*) *If* α_0 *is* ω*-homotopic to* α_1 *and* $\alpha_0 \simeq \alpha'_0$ *as maps from* (X,A,x_0) *to* $(Y,B,\omega(0))$, $\alpha_1 \simeq \alpha'_1$ *as maps from* (X,A,x_0) *to* $(Y,B,\omega(1))$, *and* $\omega \simeq \omega'$ *as paths in* B, *then* α'_0 *is* ω'*-homotopic to* α'_1.

PROOF (*a*) Given α_1 and ω, it follows from the nondegeneracy of x_0 that there is a map $H': (X,A) \times I \to (Y,B)$ such that $H'(x,0) = \alpha_1(x)$ and $H'(x_0,t) = \omega(1-t)$. Define $\alpha_0: (X,A,x_0) \to (Y,B,\omega(0))$ by $\alpha_0(x) = H'(x,1)$. Then $H: (X,A) \times I \to (Y,B)$ defined by $H(x,t) = H'(x, 1-t)$ is an ω-homotopy from α_0 to α_1.

(*b*) Because x_0 is a nondegenerate base point, there is a retraction

r: $(X,A) \times I \to (x_0 \times I \cup X \times 1, x_0 \times I \cup A \times 1)$ (by theorem 2.8.1), and we let r_t: $(X,A) \to (x_0 \times I \cup X \times 1, x_0 \times I \cup A \times 1)$ be defined by $r_t(x) = r(x,t)$. Let G: $(x_0 \times I \cup X \times 1, x_0 \times I \cup A \times 1) \times I \to (X,A) \times I$ be the homotopy relative to $(x_0,0)$ defined by $G(x,t,t') = (x,tt')$ and define

$$G_{t'}: (x_0 \times I \cup X \times 1, x_0 \times I \cup A \times 1) \to (X,A) \times I$$

by $G_{t'}(x,t) = G(x,t,t')$. Then $G_0 \circ r_0 \simeq G_1 \circ r_0$ rel x_0, and because $G_0(x_0 \times I) = (x_0,0)$, $G_0 \circ r_0 \simeq G_0 \circ r_1$ rel x_0. Let H: $(X,A) \times I \to (Y,B)$ be an ω-homotopy from α_0 to α_1. Then $H \circ G_1 \circ r_0 \simeq H \circ G_0 \circ r_1$ rel x_0. Clearly, $H \circ G_0 \circ r_1 = \alpha_0$, and so $\alpha_0 \simeq H \circ G_1 \circ r_0$ rel x_0. Similarly, if H': $(X,A) \times I \to (Y,B)$ is an ω-homotopy from α_0' to α_1, then $\alpha_0' \simeq H' \circ G_1 \circ r_0$ rel x_0. Because

$$H \mid (x_0 \times I \cup X \times 1) = H' \mid (x_0 \times I \cup X \times 1)$$

$H \circ G_1 \circ r_0 = H' \circ G_1 \circ r_0$, and so $\alpha_0 \simeq \alpha_0'$ rel x_0.

(*c*) First we observe that the inclusion map

$$(X \times \dot{I} \cup x_0 \times I, A \times \dot{I} \cup x_0 \times I) \subset (X,A) \times I$$

is a cofibration. In fact, let h: $(I \times I, I \times 0 \cup \dot{I} \times I) \approx (I \times I, 0 \times I)$ be a homeomorphism. Then there is a homeomorphism

$$1 \times h: (X \times I \times I, A \times I \times I) \approx (X \times I \times I, A \times I \times I)$$

which maps

$$X \times I \times 0 \cup X \times \dot{I} \times I \cup x_0 \times I \times I \quad \text{to} \quad X \times 0 \times I \cup x_0 \times I \times I$$

and

$$A \times I \times 0 \cup A \times \dot{I} \times I \cup x_0 \times I \times I \quad \text{to} \quad A \times 0 \times I \cup x_0 \times I \times I.$$

Thus we need only show that $(X \times 0 \cup x_0 \times I, A \times 0 \cup x_0 \times I) \times I$ is a retract of $(X \times I, A \times I) \times I$, which follows from the fact that $(X \times 0 \cup x_0 \times I, A \times 0 \cup x_0 \times I)$ is a retract of $(X \times I, A \times I)$.

Now let F, F': $(X \times \dot{I} \cup x_0 \times I, A \times \dot{I} \cup x_0 \times I) \to (Y,B)$ be defined by

$$\begin{array}{lll} F(x,0) = \alpha_0(x) & F(x,1) = \alpha_1(x) & F(x_0,t) = \omega(t) \\ F'(x,0) = \alpha_0'(x) & F'(x,1) = \alpha_1'(x) & F'(x_0,t) = \omega'(t) \end{array}$$

Because $\alpha_0 \simeq \alpha_0'$, $\alpha_1 \simeq \alpha_1'$, and $\omega \simeq \omega'$, it follows that $F \simeq F'$. Because α_0 is ω-homotopic to α_1, F can be extended to a map H: $(X,A) \times I \to (Y,B)$. By the cofibration property established above, F' can be extended to a map H': $(X,A) \times I \to (Y,B)$. Then H' is an ω'-homotopy from α_0' to α_1'. ∎

It follows from lemmas 2*a* and 2*b* that, given ω and α_1: $(X,A,x_0) \to (Y,B,\omega(1))$, the set of all maps α_0: $(X,A,x_0) \to (Y,B,\omega(0))$ which are ω-homotopic to α_1 belong to a single homotopy class of maps $(X,A,x_0) \to (Y,B,\omega(0))$. It follows from lemma 2*c* that this set of maps equals a homotopy class of maps $(X,A,x_0) \to (Y,B,\omega(0))$ which depends only on the homotopy class $[\alpha_1] \in [X,A,x_0; Y,B,\omega(1)]$ and the path class $[\omega]$. Therefore, if (X,A) has a nondegenerate base point, there is a map

$$h_{[\omega]}\colon [X,A,x_0;\ Y,B,\omega(1)] \to [X,A,x_0;\ Y,B,\omega(0)]$$

characterized by the property $h_{[\omega]}[\alpha_1] = [\alpha_0]$ if and only if α_0 is ω-homotopic to α_1. It follows from lemmas 1*a* and 1*b* that this map is functorial in (X,A) and in (Y,B) and from lemma 1*c* that if (X,A) is a suspension, the map is a homomorphism.

3 THEOREM *Let (X,A) be a pair with nondegenerate base point x_0. For any pair (Y,B) there is a covariant functor from the fundamental groupoid of B to the category of pointed sets which assigns to a point $y_0 \in B$ the set $[X,A,x_0;\ Y,B,y_0]$ and to a path class $[\omega]$ in B the map $h_{[\omega]}$. If (X,A) is a suspension, this functor takes values in the category of groups and homomorphisms.*

PROOF We need only verify the two functorial properties. If $\alpha\colon (X,A,x_0) \to (Y,B,y_0)$ is arbitrary and ε is the constant path at y_0, the constant homotopy is an ε-homotopy from α to α proving that $h_{[\varepsilon]} = 1$.

Given paths ω and ω' in B such that $\omega(1) = \omega'(0)$, an ω-homotopy H from α_0 to α_1, and an ω'-homotopy H' from α_1 to α_2 [where α_0, α_1, α_2 are maps of (X,A) to (Y,B)], an $(\omega * \omega')$-homotopy $H * H'$ from α_0 to α_2 is defined by

$$(H * H')(x,t) = \begin{cases} H(x,2t) & 0 \le t \le \frac{1}{2} \\ H'(x,\ 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

This shows that $h_{[\omega * \omega']} = h_{[\omega]} \circ h_{[\omega']}$. ■

4 COROLLARY *If $B \subset Y$ is path connected and (X,A) has a nondegenerate base point x_0, then for any y_0, $y_1 \in B$ the pointed sets $[X,A,x_0;\ Y,B,y_0]$ and $[X,A,x_0;\ Y,B,y_1]$ are in one-to-one correspondence. Furthermore, $\pi_1(B,y_0)$ acts as a group of operators on the left on $[X,A,x_0;\ Y,B,y_0]$, and the one-to-one correspondence above is determined up to this action of $\pi_1(B,y_0)$.*

PROOF If $[\omega]$ is any path class in B, $h_{[\omega]}$ is a one-to-one correspondence. If $[\omega] \in \pi_1(B,y_0)$, then $h_{[\omega]}$ is a permutation of $[X,A,x_0;\ Y,B,y_0]$, and in this way $\pi_1(B,y_0)$ acts as a group of operators. If y_0 and y_1 are points in B, the set of one-to-one correspondence $h_{[\omega]}$ determined by path classes $[\omega]$ in Y from y_0 to y_1 is the same as the set of maps $h_{[\omega_0]} \circ h_{[\omega']}$, where $[\omega_0]$ is a fixed path class from y_0 to y_1 and $[\omega'] \in \pi_1(B,y_0)$. ■

In all of the above, by taking $B = Y$, we get the corresponding results for the absolute case. Thus, if X is a space with nondegenerate base point x_0 and $y_0 \in Y$, then $\pi_1(Y,y_0)$ acts as a group of operators on $[X,x_0;\ Y,y_0]$. If Y is path connected and y_0, $y_1 \in Y$, then $[X,x_0;\ Y,y_0]$ and $[X,x_0;\ Y,y_1]$ are in one-to-one correspondence by a bijection determined up to the action of $\pi_1(Y,y_0)$.

In case Y is an H space and $B \subset Y$ is a sub-H-space, there is the following result, which can be regarded as a generalization of theorem 1.8.4.

5 THEOREM *Let (X,A) have a nondegenerate base point x_0 and let (Y,B) be a pair of H spaces. If $y_0 \in B$ is the base point, $\pi_1(B,y_0)$ acts trivially on $[X,A,x_0;\ Y,B,y_0]$.*

PROOF Let μ: $(Y \times Y,\ B \times B) \to (Y,B)$ be the multiplication. Given α: $(X,A,x_0) \to (Y,B,y_0)$ and a closed path ω: $(I,\dot{I}) \to (B,y_0)$, define an ω'-homotopy H: $(X,A) \times I \to (Y,B)$ from α' to α' (where $\omega' \simeq \omega$ and $\alpha' \simeq \alpha$) by

$$H(x,t) = \mu(\alpha(x),\omega(t))$$

Therefore $h_{[\omega]}[\alpha] = [\alpha]$ for all $[\alpha] \in [X,A,x_0;\ Y,B,y_0]$ and all $[\omega] \in \pi_1(B,y_0)$. ∎

There is an interesting relation between the action of $\pi_1(B,y_0)$ on $[X,A,x_0;\ Y,B,y_0]$ and the action of $\pi_1(B,y_0)$ as covering transformations on a universal covering space of B. We assume that B and Y are path connected and locally path connected, that $\pi_1(B,y_0) \approx \pi_1(Y,y_0)$, and that $\tilde{Y}$ is a simply connected covering space of Y with covering projection p: $\tilde{Y} \to Y$. Then $\tilde{B} = p^{-1}(B)$ is a simply connected covering space of B [because $\pi_1(B,y_0) \approx \pi_1(Y,y_0)$]. Let $\tilde{y}_0 \in p^{-1}(y_0)$. There is a canonical map

$$\theta\colon [X,A,x_0;\ \tilde{Y},\tilde{B},\tilde{y}_0] \to [X,A;\ \tilde{Y},\tilde{B}]$$

from base-point-preserving homotopy classes to free homotopy classes. Because $\tilde{B}$ is simply connected, this map is a bijection [recall that two maps α_0, α_1: $(X,A) \to (Y,B)$ are freely homotopic if and only if there is a path ω in B from $\alpha_0(x_0)$ to $\alpha_1(x_0)$ such that α_0 is ω-homotopic to α_1].

6 LEMMA *With the notation above, let g: $(\tilde{Y},\tilde{B},\tilde{y}_0) \to (\tilde{Y},\tilde{B},\tilde{y}_1)$ be a covering transformation and let $\tilde{\omega}$ be a path in $\tilde{B}$ from $\tilde{y}_0$ to $\tilde{y}_1$. There is a commutative diagram*

$$\begin{array}{ccccc}
[X,A,x_0;\ Y,B,y_0] & \xleftarrow{p_\#} & [X,A,x_0;\ \tilde{Y},\tilde{B},\tilde{y}_0] & \xrightarrow[\approx]{\theta} & [X,A;\ \tilde{Y},\tilde{B}] \\
h_{p_\#[\tilde{\omega}]}\downarrow\approx & & \approx\downarrow h_{[\tilde{\omega}]} \circ g_\# & & \approx\downarrow g_\# \\
[X,A,x_0;\ Y,B,y_0] & \xleftarrow{p_\#} & [X,A,x_0;\ \tilde{Y},\tilde{B},\tilde{y}_0] & \xrightarrow[\approx]{\theta} & [X,A;\ \tilde{Y},\tilde{B}]
\end{array}$$

PROOF Because g is a covering transformation, $p = p \circ g$ and $p_\# = p_\# \circ g_\#$. The commutativity of the left-hand square follows from this and from lemma 1*b*. Since $\theta \circ h_{[\tilde{\omega}]} = \theta$, the commutativity of the right-hand side follows from the trivial verification that $\theta \circ g_\# = g_\# \circ \theta$. ∎

Recall the isomorphism ψ: $G(\tilde{B} \mid B) \approx \pi_1(B,y_0)$ of corollary 2.6.4, which assigns to g the element $[p \circ \tilde{\omega}] \in \pi_1(B,y_0)$. Therefore lemma 6 expresses a relation between the action of $G(\tilde{B} \mid B) \approx G(\tilde{Y} \mid Y)$ on the free homotopy classes $[X,A;\ \tilde{Y},B]$ and the action of $\pi_1(B,y_0)$ on $[X,A,x_0;\ Y,B,y_0]$.

7 COROLLARY *Let X be a simply connected locally path-connected space with nondegenerate base point and let $\tilde{Y}$ be a simply connected covering space of a locally path-connected space Y. There is a bijection from the free homotopy classes $[X;\tilde{Y}]$ to the pointed homotopy classes $[X,x_0;\ Y,y_0]$ compatible with the action of $G(\tilde{Y} \mid Y)$ on the former, the action of $\pi_1(Y,y_0)$ on the latter, and the isomorphism ψ: $G(\tilde{Y} \mid Y) \approx \pi_1(Y,y_0)$.*

PROOF This follows from lemma 6, with $B = Y$ and $A = X$, and from the observation that because X is simply connected, it follows from the lifting theorem 2.4.5, the homotopy lifting property of $p: \tilde{Y} \to Y$, theorem 2.2.3, and the unique-lifting property, theorem 2.2.2, that $p_\#: [X,x_0; \tilde{Y},\tilde{y}_0] \to [X,x_0; Y,y_0]$ is a bijection. ■

We now specialize to the homotopy groups. Because

$$\pi_n(X,x_0) = [S^n(\dot{I}),0; X,x_0] = [S^n(\dot{I}),S^n(\dot{I}),0; X,X,x_0]$$

we obtain the following result.

8 THEOREM *For any space X and any $n \geq 1$, there is a covariant functor from the fundamental groupoid of X to the category of groups and homomorphisms which assigns to $x \in X$ the group $\pi_n(X,x)$ and to a path class $[\omega]$ in X the map $h_{[\omega]}: \pi_n(X,\omega(1)) \to \pi_n(X,\omega(0))$. In this way, $\pi_1(X,x_0)$ acts as a group of operators on the left on $\pi_n(X,x_0)$, by conjugation if $n = 1$, and if X is path connected and x_0, $x_1 \in X$, then $\pi_n(X,x_0)$ and $\pi_n(X,x_1)$ are isomorphic by an isomorphism determined up to the action of $\pi_1(X,x_0)$.*

PROOF Everything follows from theorem 3 and corollary 4 except for the statement that $\pi_1(X,x_0)$ acts on $\pi_1(X,x_0)$ by conjugation. For this let $H: S(\dot{I}) \times I \to X$ be on ω-homotopy from α_0 to α_1, where ω, α_0, and α_1 are closed paths in X at x_0. Define $H': I \times I \to X$ by

$$H'(t,t') = H([1,t], t')$$

Then $H' \mid 0 \times I = H' \mid 1 \times I = \omega$ and $H' \mid I \times 0 = \alpha_0$ and $H' \mid I \times 1 = \alpha_1$. It follows from lemma 1.8.6 that $(\omega * \alpha_1) * (\omega^{-1} * \alpha_0{}^{-1})$ is null homotopic. Therefore $[\alpha_0] = [\omega][\alpha_1][\omega]^{-1}$, and so $h_{[\omega]}[\alpha_1] = [\omega][\alpha_1][\omega]^{-1}$. ■

Theorem 8 shows that the action of $\pi_1(X,x_0)$ on itself by conjugation, as in theorem 1.8.3, is extended to an action of $\pi_1(X,x_0)$ on $\pi_n(X,x_0)$ for every $n \geq 1$.

A path-connected space X is said to be *n-simple* (for $n \geq 1$) if for some $x_0 \in X$ (and hence all base points $x \in X$) $\pi_1(X,x_0)$ acts trivially on $\pi_n(X,x_0)$. Thus a simply connected space is n-simple for every $n \geq 1$, and a path-connected space X is 1-simple if and only if $\pi_1(X,x_0)$ is abelian. For n-simple spaces there is a unique canonical isomorphism $\pi_n(X,x_0) \approx \pi_n(X,x_1)$, any map $\alpha: S^n \to X$ determines a unique element of $\pi_n(X,x_0)$ (whether or not α maps the base point $p_0 \in S^n$ to x_0), and $\pi_n(X,x_0)$, is in one-to-one correspondence with the free homotopy classes of maps $S^n \to X$. The latter is a useful property, and for n-simple spaces X we shall usually omit the base point and merely write $\pi_n(X)$. From theorem 5 we obtain the following generalization of theorem 1.8.4.

9 THEOREM *A path-connected H space is n-simple for every $n \geq 1$.* ■

Similar consideration apply to the relative homotopy groups.

10 THEOREM *For any pair (X,A) and any $n \geq 1$ there is a covariant functor from the fundamental groupoid of A to the category of pointed sets if $n = 1$ and the category of groups if $n > 1$ which assigns $\pi_n(X,A,x)$ to $x \in A$ and to a path class $[\omega]$ in A the map*